Statistical Physics
Fall 2024
Emilie Huffman, Subhayan Sahu

Tutorial 3

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Instructions

- We encourage you to work in groups on the problems. Groups of 3 will be assigned at random at the beginning of the tutorial
- Unless otherwise specified, you can go through the tutorial in whichever order you prefer
- If you are unable to attend a tutorial, you should work on the tutorial on your own (for about 1.5 hours) and submit your written work

1 Critical exponents of the van der Waals gas

In this question we will compute the critical exponents for the van der Waals gas (i.e., you will look at the asymptotic behaviour near the critical point given by $T = T_c, P = P_c$, and $V = V_c$). To make things easier, we will work with the reduced variables

$$P_r = \frac{P}{P_c}, \quad v_r = \frac{v}{v_c}, \quad T_r = \frac{T}{T_c}, \tag{1}$$

as well as

$$t = T_r - 1. \tag{2}$$

We will start by assuming the form of the equation of state,

$$P_r = \frac{8T_r}{3v_r - 1} - \frac{3}{v_r^2}. \tag{3}$$

You will obtain this form of the equation of state in the first homework.

(a) First let's look at the behaviour of v_r as a function of P_r when $T_r = 1$, close to the critical point. Show that

$$\delta_v = v_r - 1 \propto (\underbrace{P_r - 1}_{\delta P})^{1/\delta} \tag{4}$$

with $\delta = 3$.

(b) Next we'll look at the isothermal compressibility

$$\kappa_T = -\frac{1}{v}\frac{\partial v}{\partial P} \tag{5}$$

when $v_r = 1$ and $T_r \to 1^+$. Show that

$$\kappa_T \sim t^{-\gamma}, \quad (t \to 0^+) \tag{6}$$

with $\gamma = 1$.

Hint: look at $\left.\frac{\partial P_r}{\partial v_r}\right|_{v_r=1}$ first.

(c) Next we'll see how v_l and v_g depend on T_r as $T_r \to 1^-$. To make this simpler, we'll write

$$v_l = 1 - \delta v_l, \quad v_g = 1 + \delta v_g, \tag{7}$$

where the corrections are small and positive.

Using this and the fact that

$$\frac{8T_r}{3v_l - 1} - \frac{3}{v_l^2} = \frac{8T_r}{3v_g - 1} - \frac{3}{v_g^2} \tag{8}$$

show that

$$v_l - 1 \sim -|2t|^\beta, \quad v_g - 1 \sim |2t|^\beta \quad (t \to 0^-) \tag{9}$$

with $\beta = 1/2$.

(d) Last we'll look at the behaviour of the specific heat at constant volume,

$$C_V = \left(\frac{\partial U}{\partial T}\right)_V, \tag{10}$$

where it's important to note that we are thinking of U as a function of (T, V) rather than (S, V). N is also constant.

- First let's find the expression for $U(T,V) = F + TS$. Start by proving that

$$\left(\frac{\partial U}{\partial V}\right)_T = -P + T\left(\frac{\partial P}{\partial T}\right)_V \tag{11}$$

- Next show that

$$U(T,v) = U_0(T) + N\int_{\infty}^{v}\left(\frac{\partial U}{\partial V}\right)_T dv \tag{12}$$

  and argue on physical grounds that $U_0(T) = \frac{3}{2}NkT$.

- At this point you should have

$$U(T,v) = \frac{3}{2}NkT - \frac{aN}{v} \tag{13}$$

  Show that

$$C_V \sim \frac{3}{2}Nk|t|^{-\alpha} \quad (t \to 0) \tag{14}$$

  with $\alpha = 0$.

2 Critical exponents from Landau theory

In Landau theory for problems related to magnetization, the partition function is expressed as a functional integral of an n component real valued order parameter field $\vec{m}(x)$ existing in a d-dimensional space; where $\vec{x} \equiv (x_1, x_2, \cdots, x_d)$ and $\vec{m} \equiv (m_1, m_2, \cdots, m_n)$, and a Landau Ginzburg 'Hamiltonian' for the order parameter field, $\beta\mathcal{H}\left[\vec{m}(\vec{x})\right]$.

Some specific cases covered in this general framework (with $d = 3$):

$n = 1$ describes liquid–gas transitions, binary mixtures, as well as uniaxial magnets;

$n = 2$ applies to superfluidity, superconductivity, and planar magnets;

$n = 3$ corresponds to classical magnets.

In this framework, we don't worry about the microscopic origins of the Hamiltonian, and instead write down any Hamiltonian terms consistent with the global symmetry, $\vec{m} \to R_n\vec{m}$, with R_n being the rotation in the $n-$dimensional order parameter space. The partition function is written as,

$$Z = \int \mathcal{D}\vec{m} \exp\left(-\beta\mathcal{H}[\vec{m}]\right). \tag{15}$$

(a) What is the symmetry group associated with the one component, and the two component cases? Are the groups discrete or continuous?

(b) In the absence of an external field, show that the lowest order terms in the Landau Ginzburg Hamiltonian are given by,

$$\beta\mathcal{H} = \beta F_0 + \int d^d\vec{x}\left[\frac{t}{2}m^2(\vec{x}) + um^4(\vec{x}) + \frac{K}{2}(\nabla m)^2 + \cdots\right] \tag{16}$$

Why can we ignore terms such as ∇m or $\vec{m}(\vec{x})^3 \equiv \sum_i m_i(\vec{x})m_i(\vec{x})m_i(\vec{x})$?

Now assume that we have an added contribution from the magnetic work $\vec{B}\cdot\vec{m}$ to the Hamiltonian, where $\vec{h} \equiv \beta\vec{B}$ and $\vec{B}$ is the external magnetic field. How is the Landau Ginzburg Hamiltonian modified? [1]

(c) In the saddle point approximation, the integral can be replaced by the maximum value of the integrand. Is the saddle point free energy an analytic function of the parameters $t, u, K \cdots$?

As a first pass, let us assume the saddle points are obtained by uniform solutions of the field, $\vec{m}(\vec{x}) \equiv \vec{m}$. When is the saddle point approximation justified? We assume that the minimizing order parameter field is along the same direction as the external field for the further computations.

What is the function $\Psi(\vec{m})$ which is to be extremized to obtain the saddle point solutions, when we keep only upto the quartic term in the Landau Ginzburg Hamiltonian? What should be the value of u such that the integral is well defined?

(d) Plot the following functions

- $\Psi(\vec{m})$ as a function of m (magnitude of $\vec{m}$), assuming $t > 0$, and for different h, namely $h < 0$, $h = 0$, and $h > 0$.
- $\Psi(\vec{m})$ as a function of m, assuming $t < 0$, and for different h, namely $h < 0$, $h = 0$, and $h > 0$.
- The minima $\overline{m}$ as a function of h, for $t > / = / < 0$.

(e) Identify the first order and second order transitions from the above analysis.

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[1] This technically modifed the ensemble from being canonical to the Gibbs canonical; however we will still (misleadingly) denote the corresponding free energy as F, which is usually reserved for the Helmholtz free energy.

(f) We will minimally identify $t = a(T - T_c)$ as the monotonic function of temperature, for a non-universal function $a > 0$.

- At $h = 0$ (zero field), find the expressions for the saddle point magnetization, $\overline{m}$, longitudinal susceptibility $\chi_l = \frac{\partial \overline{m}}{\partial h}\big|_{h=0}$, heat capacity, $C(h = 0) = -T\frac{\partial^2 F}{\partial T^2}$ as a function of $T_c - T$.
- At the critical isotherm, $t = 0$, find the expression for the magnetization $\overline{m}(t = 0)$ as a function of h.

Identify the corresponding critical exponents, $\alpha, \beta, \gamma, \delta$. Recall, $C \sim t^{-\alpha}$, $\overline{m}(h = 0) \propto t^{\beta}$, $\chi \sim t^{-\gamma}$, $\overline{m}(t = 0) \propto h^{1/\delta}$.