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Tutorial 4

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1 Multicritical points in Landau theory

Tricritical point: By tuning an additional parameter, a second order transition can be made first order. The special point separating the two types of transitions is known as a tricritical point, and can be studied by examining the Landau–Ginzburg Hamiltonian

$$\beta\mathcal{H} = \int d^d x \underbrace{\left[\frac{K}{2}(\nabla m)^2 + \frac{t}{2}m^2 + um^4 + vm^6 - hm\right]}_{\Psi(m)}$$

where u can be positive or negative. For $u < 0$, a positive v is necessary to insure stability.

(a) By sketching the energy density $\Psi(m)$, for various t, show that in the saddle point approximation there is a first order transition for $u < 0$ and $h = 0$.

(b) Calculate $\bar{t}$ and the discontinuity $\overline{m}$ at this transition.

(c) For $h = 0$ and $v > 0$, plot the phase boundary in the (u, t) plane, identifying the phases, and order of the phase transitions.

(d) The special point $u = t = 0$, separating first and second order phase boundaries, is a tricritical point. For $u = 0$, calculate the tricritical exponents β, δ, γ and α, governing the singularities in magnetization, susceptibility, and heat capacity. (Recall, $C \sim t^{-\alpha}$, $\overline{m}(h = 0) \propto t^{\beta}$, $\chi \sim t^{-\gamma}$, $\overline{m}(t = 0) \propto h^{1/\delta}$.)

(e) Note that the correlation length ξ is related to the curvature $\Psi(m)$ at its minimum by $K\xi^{-2} = \partial^2\Psi/\partial m^2|_{\text{eq}}$.

To see this is the case, consider the fluctuation around the uniform saddle, $m(x) = m_{\text{eq}} + \phi(x)$ for $\phi(x) \ll m_{\text{eq}}$. Expand the energy density can be expanded around the equilibrium value to identify that the correlation length scale is indeed the expression given above.

What is the limit of ξ along the first and second order transitions?

2 Scaling theory

In this question you'll prove that the scaling hypothesis for the singular part of the Landau free energy,

$$f_s(t, H) = |t|^{2-\alpha} g(H|t|^{-\Delta}) \tag{1}$$

relates the critical exponents β, γ, δ to the α and Δ, so that only two of them are independent.

(a) Show that $M(t, H)$ is a homogeneous function and that $\beta = 2 - \alpha - \Delta$.

(b) Show that $\chi(t, H)$ is a homogeneous function and that $-\gamma = 2 - \alpha - 2\Delta$.

(c) Show that at $t = 0$ we have

$$M(0, H) \propto \operatorname{sgn}(H)|H|^{\beta/\Delta} \quad (H \to 0) \tag{2}$$

so $\delta = \Delta/\beta$.

Hint: For part (c) start by convincing yourself that in order to have the phase transition the limit

$$\lim_{t \to 0} |t|^{\beta} g'(H|t|^{-\Delta}) \neq 0 \tag{3}$$

when $H \neq 0$.