Tutorial 5

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1 Block Spin RG for 1D Ising model

Let's look at the Ising model with Ising variables $\sigma_i = \pm 1$ on a 1D lattice on a periodic boundary condition $\sigma_{N+1} \equiv \sigma_1$. In this model we will carry out the RG procedure exactly. We will assume that N is even, and the Hamiltonian is given by,

$$H = -J \sum_i \sigma_i \sigma_{i+1} - \sum_i h \sigma_i$$

and consider the case $h = 0$. Note, we can introduce the combined parameter $K = \beta J$.

(a) Show that the partition function $Z[K]$ can be written as a trace of a product of transfer matrices, $Z[K] = \sum_{\sigma_1, \sigma_2, \cdots, \sigma_N} V_{\sigma_1 \sigma_2} V_{\sigma_2 \sigma_3} \cdots V_{\sigma_N \sigma_1}$. Write down the expression for the entries of the transfer matrix.

(b) The specific averaging procedure that we use in this case is known as *decimation*, and consists in summing over only half of the spins, say all the even ones. By renaming the spin $s_i = \sigma_{2i-1}$, express the partition function $Z[K]$ as a product of transfer matrices of the form $V^2_{s_1 s_2}$.

Next make the r.h.s. look like the original partition function, but with different parameters and half the sites, and a different transfer matrix $V'_{s_1 s_2}$ (with a different parameter K') of the same form as V. Write down the relation by identifying the renormalized coupling K' as a function of the old K, and a function $a(K')$ that establishes the relation between the transfer matrices,

$$V' \equiv V[K]^2 = a(K') V[K']. \tag{1}$$

.

Comment: At the end of this process, we have the equality between the partition functions, $Z(N,K) = Z(N/2,K')$. This procedure only works exactly in 1D: in higher dimensions the decimation procedure is going to introduce interactions between sites that are not nearest neighbours.

(c) What are the fixed points in the transformation $K \to K'$? Determine the stability of the fixed points.

(d) We identify the $K^* = \infty$ fixed point (zero Temperature) as the second order phase transition. We will now use the RG equation to derive the critical exponents ν and η.

First, note that the the correlation function $g_{ij}(K')$ defined as $\langle s_i s_j \rangle_{K'}$ equals $\langle \sigma_{2i-1}\sigma_{2j-1}\rangle_K$.

Assuming that

$$g_{ij}(K) \sim \begin{cases} |i-j|^{-\tau} e^{-|i-j|/\xi(K)} & \text{if } K < \infty \\ |i-j|^{-d+2-\eta} & \text{if } K = \infty \end{cases} \tag{2}$$

with $d = 1$, prove that $\eta = 1$. Now, for $T > 0$, show that the correlation length $\xi \to \infty$ at $K \to \infty$ as $\xi = 1/t$, for $t = e^{-2K}$.

Hint: Away from the scale invariant point, think about how the correlation length changes under rescaling. It is convenient to introduce a new parameter $x = \tanh K$ and rewrite the RG equation in terms of this parameter. Now using these two ideas, deduce the scaling of the correlation length ξ as a function of x and eventually K.

2 Scaling theory from RG

In the last tutorial you found the relations between the different critical exponents arising from the scaling hypotheses. Now, we will derive how the scaling hypothesis arises from RG.

From the last problem, we found that the RG step is a transformation in the space of the parameters of the Hamiltonian. A single RG step is as follows:

1. Coarse grain by scale b:

$$m(x) = \frac{1}{b^d}\int_{\text{ball of size } b \text{ centered around } x} dx' m(x')$$

.

2. Rescale distances, $x_{\text{new}} = x_{\text{old}}/b$.

3. Renormalize the order parameter (if the 'contrast' changes)

$$m_{\text{new}}(x_{\text{new}}) = \frac{1}{\zeta b^d} \int_{\text{ball of size } b \text{ centered around } bx_{\text{new}}} dx' m_{\text{old}}(x'_{\text{old}})$$

4. Modify the Hamiltonian parameters to obtain the probabilities of configurations $m_{\text{new}}(x_{\text{new}})$ as, $e^{-\beta \mathcal{H}_b[m_{\text{new}}(x_{\text{new}})]}$.

Suppose we have a two parameter RG flow depending on the parameters t, h in the Hamiltonian (think about the reduced temperature and the external field for the Ising model).

Around the critical point $t, h = 0$, the modified parameters t' and h' can be expanded to its linear order as,

$$\begin{aligned} t' &\equiv t_b = b^{y_t} t + \cdots \\ h' &\equiv h_b = b^{y_h} h + \cdots \end{aligned} \tag{3}$$

We can now explore some consequences of the perturbative RG equation around criticality.

(a) Generically, we should have obtained a Taylor expanded perturbative RG equations:

$$\begin{aligned} t' &\equiv t_0 + A(b)t + B(b)h + \mathcal{O}(t^2, th, h^2) \\ h' &\equiv h_0 + C(b)t + D(b)h + \mathcal{O}(t^2, th, h^2) \end{aligned}$$

Why can we ignore the zeroth order and other first order terms in the above perturbative RG equations?

(b) What should be the sign of y_t and y_h?

(c) Starting from the identification $Z = Z'$, and therefore the equality of the Free energies before and after the RG step, show that the free energy density must be a homogenous function,

$$f(t, h) = b^{-d} f(b^{y_t} t, b^{y_h} h) \tag{4}$$

Next, identify the the exponents α and the gap exponent Δ in the scaling hypothesis by comparing this against the scaling hypothesis form.

$f_s(t, H) = |t|^{2-\alpha} g(H|t|^{-\Delta})$.

(d) Correlation length: All length scales are reduced by a factor of b during the RG transformation. This is also true of the correlation length. How is the correlation length exponent ν related to the RG exponents y_t and y_h? Does this imply a relation between ν, d, and α?

(e) By computing the magnetization by assuming the homogenous form of the free enrgy, obtain the critical exponent β in terms of y_h and y_t.

Comment: Once you solve this problem, it should be apparent that quite generally, the singular part of any quantity X has a homogeneous form

$$X(t,h) = b^{y_X} X(b^{y_t} t, b^{y_h} h).$$

For any conjugate pair of variables, contributing a term $\int d^d x F \cdot X$, to the Hamiltonian, the scaling dimensions are related by $y_X = y_F - d$, where $F' = b^{y_F} F$ under RG.