Emilie Huffman, Subhayan Sahu

Tutorial 6

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Gaussian model for $T < T_c$

The Gaussian model (for the Ising transition) you have seen in the lecture notes only makes sense for $T > T_c$. Here you will derive a version of it that works on the other side of the phase transition as well.

We'll work with $H = 0$ for the sake of simplicity, but the approach can be easily generalised. Let's start with the exact partition function

$$Z = \sqrt{\det\left(\frac{2\beta A^2 B}{\pi}\right)} \int_{\mathbb{R}^N} \mathrm{d}^N\phi \, e^{-S(\phi)} \tag{1}$$

with

$$S(\phi) = \frac{\beta A^2}{2}\phi^{\mathrm{t}} B\phi - \sum_i \ln(\cosh(\beta A(B\phi)_i)), \tag{2}$$

where A is a number that relates the field ϕ to the Ising variable, $\langle\sigma_i\rangle \sim A\langle\phi_i\rangle$, and B is the Ising interaction matrix in real space, with $B_{ij} = J_{ij} + \lambda\delta_{ij}$, as described in the notes. You can think of ϕ as a real valued N component column vector.

The goal is to evaluate the integral using the saddle-point approximation, i.e.,

$$S(\phi) \approx S(\psi) + \frac{1}{2}\sum_{i,j}(\phi-\psi)_i(\phi-\psi)_j \frac{\partial^2 S}{\partial\phi_i \, \partial\phi_j}(\psi) \tag{3}$$

where ψ minimizes S, making Z a Gaussian integral.

1. First of all, show that

$$\sum_j B_{ij} = B_0 \tag{4}$$

using the lattice Fourier transform of B. As we have done in the lectures, we will interpret B_0 as the critical temperature, i.e.,

$$B_0 = k_\mathrm{B} T_c. \tag{5}$$

2. We will look for a ψ that is the same for every lattice site, i.e.,

$$\psi_i = \bar{\psi}. \tag{6}$$

Show that, if we introduce $M = A\bar{\psi}$, ψ is a stationary point of S if and only if

$$M = \tanh\left(\frac{T_c M}{T}\right), \tag{7}$$

which is the equation from mean field theory! In particular, you already know how M behaves near $T = T_c$.

3. Show that, as long as T is close to T_c, the Hessian matrix $\mathcal{H} = \frac{\partial^2 S}{\partial\phi_i\,\partial\phi_j}(\psi)$ is positive definite, making ψ a minimum point (and the Gaussian integral well-defined). **Hint:** Use the fact that

$$\operatorname{sech}(\operatorname{arctanh}(M)) = \sqrt{1 - M^2}. \tag{8}$$

4. Write the approximate expression for $S(\phi)$ on both sides of the phase transition. Does the result match the mean field theory result that you already knew from the lecture notes?

5. Compute the partition function in the new Gaussian approximation you found.

6. Compute the average spin

$$\langle \sigma_i \rangle = A\,\langle \phi_i \rangle_S \approx A \frac{\int_{\mathbb{R}^N} \mathrm{d}^N\phi\, e^{-\frac{1}{2}(\phi-\psi)^\mathrm{t}\partial^2 S(\psi)(\phi-\psi)}\phi_i}{\int_{\mathbb{R}^N} \mathrm{d}^N\phi\, e^{-\frac{1}{2}(\phi-\psi)^\mathrm{t}\partial^2 S(\psi)(\phi-\psi)}}. \tag{9}$$

What do you notice about the result? What is the critical exponent β?