Statistical Physics
Fall 2024
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Tutorial 7

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1 Dimensional Analysis

From Set 1 of the week 4 notes, we saw that the ϕ^4 action in momentum space is

$$S(\varphi) = \frac{1}{2V}\sum_k (r+k^2)|\varphi_k|^2 - h\varphi_0 + \frac{u}{4!V^3}\sum_{k_1,k_2,k_3,k_4} \varphi_{k_1}\varphi_{k_2}\varphi_{k_3}\varphi_{k_4}\,\delta_{k_1+k_2+k_3+k_4,0}. \tag{1}$$

1. Starting from $[x]=1$, determine the dimensions of r, h, u, φ_k, and ϕ_i.
2. Discuss how we could have inferred the leading order behavior of the RGT for the Gaussian model using the information about the dimension of r, h, and φ.

   Recall: $\varphi' = b^{-\frac{D+2}{2}}\varphi,\ h' = b^{\frac{D+2}{2}}h,\ r' = b^2 r$ are the RG transformations for the Gaussian model.

2 Computing η and ν

In Set 1 of the week 4 notes, we compute the critical exponent α for the Gaussian theory, which is given in momentum space by

$$Z \propto \int D^N\varphi\, e^{-\frac{1}{2V}\sum_k (r+k^2)|\varphi_k|^2}. \tag{2}$$

Here we will compute η and ν for the theory, since we haven't before. The following integrals will be useful:

$$\langle \varphi_k \rangle = \frac{\int^\Lambda D^N\varphi\, e^{-\frac{1}{2V}\sum_k (r+k^2)|\varphi_k^2|}\;\varphi_k}{\int^\Lambda D^N\varphi\, e^{-\frac{1}{2V}\sum_k (r+k^2)|\varphi_k^2|}} = 0$$

$$\langle \varphi_k \varphi_q \rangle = \frac{\int^\Lambda D^N\varphi\, e^{-\frac{1}{2V}\sum_k (r+k^2)|\varphi_k^2|}\;\varphi_k\varphi_q}{\int^\Lambda D^N\varphi\, e^{-\frac{1}{2V}\sum_k (r+k^2)|\varphi_k^2|}} = \frac{V\delta_{k+q,0}}{r+k^2}. \tag{3}$$

1. To get these exponents, we need to find the real-space correlation function,

$$g(x_i - x_j) = \langle \phi_i \phi_j \rangle - \langle \phi_i \rangle \langle \phi_j \rangle\,, \tag{4}$$

where ϕ_i, ϕ_j are the fields in real-space. Show that the correlation function (as $N \to \infty$, where N is the number of lattice sites), is given by

$$g(x) = \frac{1}{(2\pi)^D}\int_{[-\Lambda,\Lambda]^D} d^D k \frac{e^{ikx}}{r+k^2}. \tag{5}$$

2. We can choose a specific direction for x, say $\vec{x} = \rho\,\vec{e}_D$, and the integration will be simpler. Argue that, as $r \to 0^+$,

$$g(x) \sim \int_{-\infty}^{\infty} dk_D \int_0^\infty dq\, q^{D-2} \frac{e^{ik_D\rho}}{r + k_D^2 + q^2}, \tag{6}$$

where k_D is the k-component parallel to x.

3. Assuming $r > 0$ and integrating, you should get

$$g(x) \sim \frac{r^{\frac{D-2}{4}}}{|x|^{\frac{D-2}{2}}} K_{\frac{D-2}{2}}\left(|x|\sqrt{r}\right), \tag{7}$$

where $K_\alpha(z)$ is the modified Bessel function of the second kind. What is the condition on D to get this expression? You can use Mathematica for this.

Hint: Use the fact that when $a > 0$

$$\int_{-\infty}^{\infty} dk \frac{e^{ikx}}{a+k^2} = \frac{\pi e^{-\sqrt{a}|x|}}{\sqrt{a}}. \tag{8}$$

To compute the remaining integral with Mathematica it is helpful to use the substitution

$$u = \sqrt{r + q^2} \tag{9}$$

first.

4. Asymptotically we have that

$$K_{\frac{D-2}{2}}(z) \sim \frac{e^{-z}}{\sqrt{z}} \quad \text{as } z \to \infty. \tag{10}$$

Combining with (7), find ν.

5. Finally, we can find η by setting $r = 0$ in (6) and doing the integral with that condition. You can assume $D > 2$ and you should find $\eta = 0$.

3 Gaussian fixed point

We mentioned in the lecture notes that when $D > 4$ the critical point of the ϕ^4 model is described by the Gaussian fixed point $(r^*, u^*) = (0, 0)$. Let's see why that is the case and what the predicted results are.

Recall that

$$\begin{cases} r' = b^2\left(r + \dfrac{u}{2} I_1\right) \\ u' = b^{\varepsilon}\left(u - \dfrac{3u^2}{2} I_2\right) \end{cases} \tag{11}$$

where $\varepsilon = 4 - D$ and

$$I_1 = \frac{K_D}{\Lambda^D} \int_{\Lambda/b}^{\Lambda} \mathrm{d}q \, \frac{q^{D-1}}{r + q^2}, \quad I_2 = \frac{K_D}{\Lambda^D} \int_{\Lambda/b}^{\Lambda} \mathrm{d}q \, \frac{q^{D-1}}{(r + q^2)^2}. \tag{12}$$

1. Linearise eq. (11) about the Gaussian fixed point. Unlike the $D < 4$ case, there will be no need to take derivatives w.r.t. b or expand in terms of ε.

   Hint: Expand the integrands in I_1 and I_2 to the appropriate order in r and then integrate.

2. Write the RGT in the form

$$\begin{pmatrix} r' \\ u' \end{pmatrix} = M(b) \begin{pmatrix} r \\ u \end{pmatrix} \tag{13}$$

and find the eigenvalues and eigenvectors of $M(b)$, as well as the associated scaling fields t_1 and t_2.

3. Compute the critical exponent ν.

4. **Discussion**: What happens when $D < 4$? What about $D = 4$?