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Tutorial 8

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1 Classical to Quantum mapping: Ising spin in a transverse field

We will see how the statistical mechanics of the Ising chain,

$$H = -K \sum_{\ell=1}^{M_\tau} \sigma_\ell^z \sigma_{\ell+1}^z - h \sum_{\ell=1}^{M_\tau} \sigma_\ell^z$$

can be mapped onto the quantum mechanics of a single Ising spin. Here M_τ number of Ising spins are placed on a 1d chain with lattice constant a.

The partition function, $Z = \sum_{\{\sigma_\ell^z\}} \exp(-H)$ can be rewritten using Transfer matrices T_1 and T_2, as

$$Z = \sum_{\{\sigma_\ell^z = \pm 1\}} \prod_{\ell=1}^{M_\tau} T_1\left(\sigma_\ell^z \sigma_{\ell+1}^z\right) T_2(\sigma_\ell^z),$$

where, $T_1\left(\sigma_1^z, \sigma_2^z\right) = \exp\left(K\sigma_1^z \sigma_2^z\right)$ and $T_2(\sigma^z) = \exp\left(h\sigma^z\right)$. Note this has a slightly different notation than the transfer matrix problem we solved during the tutorial. Recall that using the transfer matrix tool, we found that the two-point spin correlator for $h = 0$, defined as $C(\ell - \ell') \equiv \langle \sigma_\ell^z \sigma_{\ell'}^z \rangle$, for $\ell' \geq l$, was found to be,

$$C(\ell - \ell') = (\tanh K)^{\ell' - \ell}.$$

It is useful to label the spins not by the site index ℓ but by a physical length coordinate τ, by identifying,

$$\tau = \ell a,$$

and a physical length scale for the entire chain, $L_\tau = M_\tau a$.

1. Identify the correlation length from the expression above, in terms of a and K. Confirm the behavior of the correlation length as $K \to \infty$.

2. Now, we will operate in the scaling limit, $\xi \gg a$. In this limit, show that the transfer matrix can be approximately written as,

$$T_1 T_2 \approx \exp\left(-a\left(\underbrace{-\frac{K}{a}}_{E_0}-\frac{\Delta}{2}\hat{\sigma}^x-\underbrace{\frac{h}{a}}_{\tilde{h}}\hat{\sigma}^z\right)\right),$$

   where now the terms in the exponent are the Pauli x and z operators.

   Show that the partition function can now be written as $Z = \mathrm{Tr}\exp(-H_Q/T)$, for the appropriately defined quantum Hamiltonian of a single spin, at a temperature T. How are the temperature T and the parameter Δ related to the parameters in the classical Ising model?

3. Compute the free energy of the quantum spin $F = -T\ln Z$. This is the same answer that you would have obtained from the classical Ising model in the appropriate limit. This is another evidence of "universality" near the second order phase transition.

4. The correspondence between H_0 and T also extends to correlation functions. Let us define the time ordered correlator, C, of H_0 at imaginary time by

$$C(\tau_1,\tau_2) = \begin{cases} \frac{1}{2}\mathrm{Tr}\left(e^{-H_Q/T}\hat{\sigma}^z(\tau_1)\hat{\sigma}^z(\tau_2)\right) & \text{for } \tau_1 > \tau_2, \\ \frac{1}{2}\mathrm{Tr}\left(e^{-H_Q/T}\hat{\sigma}^z(\tau_2)\hat{\sigma}^z(\tau_1)\right) & \text{for } \tau_1 < \tau_2, \end{cases} \tag{6}$$

   where $\hat{\sigma}^z(\tau) \equiv e^{H_Q\tau}\hat{\sigma}^z e^{-H_Q\tau}$. Argue that this is the same as the classical model correlation function.

2 Quantum Ising chain to classical Ising model

Now we go the other way, and show that the quantum Ising model in d spatial dimension can be related to the $d+1$ classical Ising model in an appropriate scaling limit. We will specialize in the case of $d = 1$.

The 1d Quantum Ising model is defined as,

$$H_Q = -Jg\sum_i \hat{\sigma}_i^x - J\sum_{\langle ij\rangle}\hat{\sigma}_i^z\hat{\sigma}_j^z, \tag{1}$$

for nearest neighbor sites $\langle i,j\rangle$. We will use M to describe the number of qubits, with $M\to\infty$ being the thermodynamic limit. The parameter g controls the phase diagram of the Quantum Ising chain at $T=0$.

We consider the finite temperature partition function for this model, at a Temperature T,

$$Z = \mathrm{Tr}\exp\left(-H_Q/T\right). \tag{2}$$

We will now consider a transfer matrix associated with the imaginary time evolution over a short time ε, that discretizes the inverse temperature scale into M_τ discrete parts, identifying $M_\tau\varepsilon = 1/T \equiv L_\tau$. We will be interested in the scaling limit $\varepsilon\to 0$ and $M_\tau\to\infty$ while keeping $1/T$ fixed.

1. Show that in the scaling limit, the exponential of the quantum Hamiltonian can be approximated by a product of $2^M\times 2^M$ dimensional transfer matrices as $\exp(-\varepsilon H_Q)\approx T_1T_2+\mathcal{O}(\varepsilon^2)$, where T_1 involves the Pauli x operators, and T_2 involves the products of the Pauli z operators. Write down the expressions for the transfer matrices.

2. Now insert a complete set of states between each T_1T_2 term in $Z=\mathrm{Tr}\left(T_1T_2\right)^{M_\tau}$. To do this, choose the eigenstates $|\{m_i(\ell)\}\rangle$ which are the ±1 eigenstates of all $\hat{\sigma}^z_i$, for every "time" step ℓ. Then demonstrate that the quantum partition function can be reinterpreted as a partition function of an anisotropic classical Ising model,

$$Z = \sum_{\{m_i(\ell)\}}\exp\left(\sum_{i,\ell}\left[Jam_i(\ell)m_{i+1}(\ell)+Bm_i(\ell)m_i(\ell+1)\right]\right). \tag{3}$$

   Write down the form of B in terms of the Ising parameters J, g and ε.

   Hint: Use the identity $\langle m|\exp\left(Jga\hat{\sigma}^x\right)|m'\rangle = A\exp\left(Bmm'\right)$. Deduce the expressions for A and B for this to be true.

Now we consider the limit $T=0$. In this limit, we can keep ε to be small but non-zero, and take the "thermodynamic" limit $\beta\to\infty$ and still be in the scaling limit where the above derivation works. The phase diagram We can take the $\varepsilon\to 0$ limit at the end without modifying the physical results.

Notice that in this limit, the 2d classical Ising model has anisotropic couplings, $K_x = J\varepsilon$ and $K_\tau = -\frac{1}{2}\log\tanh(Jg\varepsilon)$. One can rigorously show that there is a classical phase transition in the anisotropic Ising model when g is tuned using Kramers Wannier duality that relates the high temperature and the low temperature phases of the classical Ising model, which we will not show here. However, from that argument, you can deduce that if there is a phase transition, it must occur at a point where $\sinh 2K_x = \sinh 2K_\tau$.

1. Can you deduce where the phase transition occurs using the information above, i.e. what is g_c?
2. What are the phases for $g \ll g_c$ and $g \gg g_c$?
3. Can you see why the critical behavior in the vicinity of the phase transition g_c matches the 2d classical Ising universality class?