Homework 2

Deadline: May 2. (Pass/Fail deadline May 15.)

1 Local indistinguishibility in topologically ordered ground states

Consider the toric code model on a 2d torus,

$$H = -\sum_v \left(\prod_{e \ni v} X_e\right) - \sum_p \left(\prod_{e \in p} Z_e\right),$$

which has a four dimensional ground-state subspace. Here e, v, p refer to edges, vertices, and plaquettes on the square lattice respectively. Show that any state in the ground-state subspace is locally indistinguishable.

You can prove this statement in several ways (feel free to look for your own method first!); in this problem we will guide you through a specific solution.

a) First we introduce the notation: let Γ_1 and Γ_2 be the two non-trivial cycles in the toric code lattice on the torus, and $\Gamma_1^{'}$ and $\Gamma_2^{'}$ be the two non-trivial cycles in the dual lattice, such that Γ_1 is perpendicular to $\Gamma_1^{'}$ and Γ_2 is perpendicular to $\Gamma_2^{'}$. We will refer to the string of X and Z operators along a non-trivial cycles Γ as X_Γ and $\Gamma^{'}$ as $Z_{\Gamma'}$. The logical operators are thus $X_{\Gamma_1}, X_{\Gamma_2}, Z_{\Gamma_1'}, Z_{\Gamma_2'}$ (check that these commute with the stabilizers).

   Consider $|\psi_1\rangle$ to be a ground state of the toric code that satisfies $Z_{\Gamma_1'} |\psi_1\rangle = |\psi_1\rangle$ and $Z_{\Gamma_2'} |\psi_1\rangle = |\psi_1\rangle$.

   Now show that the states $|\psi_2\rangle = X_{\Gamma_1} |\psi_1\rangle$, $|\psi_3\rangle = X_{\Gamma_2} |\psi_1\rangle$, and $|\psi_4\rangle = X_{\Gamma_1} X_{\Gamma_2} |\psi_1\rangle$ are orthogonal ground-states spanning the ground state subspace of the toric code.

b) Now consider a contractible contiguous region in the torus, labeled A. We want to show that for any state in ground state subspace, the reduced density matrix on A is the same. Show that a sufficient condition to proving this is demanding that for any local operator O_A,

$$\langle\psi_i| O_A |\psi_j\rangle = c\delta_{ij}, \tag{1}$$

for a constant c.

Hint: Consider any state in the ground-state subspace to be a superposition of the basis set defined earlier.

c) First argue that $\langle\psi_i| O_A |\psi_i\rangle = \langle\psi_j| O_A |\psi_j\rangle$ for any i, j, by checking the definitions of $|\psi_i\rangle$.

Hint: You can deform the string operators around the torus to be supported entirely on the complement of A, i.e. $\overline{A}$.

d) Next, argue that $\langle\psi_i| O_A |\psi_j\rangle = 0$ for $i \neq j$. You can do this by first choosing a specific pair, for example, $\langle\psi_1| O_A |\psi_2\rangle$. Consider the Schmidt decomposition of the states $|\psi_{\sigma=1,2}\rangle$ on the regions A and its complement $\overline{A}$, as $|\psi_\sigma\rangle = \sum_\lambda A_{\sigma,\lambda} |\psi_{\sigma,\lambda}\rangle_A |\overline{\psi}_{\sigma,\lambda}\rangle_{\overline{A}}$. By using the fact that ψ_1 and ψ_2 are defined such that their eigenvalues of the $Z_{\Gamma_1'}$ are different, argue that $\langle\overline{\psi}_{1,\lambda}|\overline{\psi}_{2,\lambda}\rangle_{\overline{A}} = 0$, thus proving the original proposition.

e) Finally let us compare this scenario to ordinary spontaneous symmetry breaking. For $\mathbb{Z}_2$ symmetry breaking in the quantum Ising model $H = -\sum_i Z_i Z_{i+1}$, the ground-state subspace has two-fold degeneracy spanned by $|GHZ_\pm\rangle \sim |000\cdots 0\rangle \pm |111\cdots 1\rangle$. Show that the reduced density matrix of either of these two states on a subregion (strictly smaller than the entire system) is the same. However, show that not all states in the ground-state subspace has this property. You can demonstrate this by constructing two specific states in the ground-state subspace that are locally distinguishable.

2 Berry Phase and Chern number

In this problem you will compute some important quantities related to the Berry phase, and use it compute electromagnetic response in Chern insulators, which exhibit behavior akin to the Integer quantum Hall effect, but without any external magnetic field!

Suppose a Hamiltonian $H(\lambda)$ depends on several parameters $\lambda \equiv (\lambda_1, \lambda_2, ...,)$, with $\{|n(\lambda)\rangle\}$ being the orthonormal eigenstates of the Hamiltonian. We want to investigate the topology of the space of eigenstates as we move around the parameter space. To study that, we can introduce a gauge connection in the space of parameters, called the Berry connection (defined for each eigenstate n),

$$\mathbf{A}_j^n(\lambda) = -i \langle n(\lambda)| \frac{\partial}{\partial \lambda_j} |n(\lambda)\rangle \tag{2}$$

This quantity is only defined when the eigenstates do not cross while the parameters are varied. The Berry connection integrated along a closed path in parameter space captures the extra phase (dubbed Berry phase) that an eigenstate picks up when the parameters are varied adiabatically along some closed path $\mathcal{C}$ in parameter space. This is referred to as he Berry phase,

$$e^{i\gamma^n} = \exp\left(-i \oint_{\mathcal{C}} \mathbf{A}^n(\lambda) \cdot d\lambda\right). \tag{3}$$

There is a (gauge) redundancy in the information contained in the Berry connection $\mathbf{A}^n(\lambda)$. This follows from the arbitrary choice we made in fixing the phase of the reference states $\{|n(\lambda)\rangle\}$. It is thus natural to compute a gauge invariant quantity, named the Berry curvature,

$$\mathcal{F}_{ij}^n = \frac{\partial A_j^n}{\partial \lambda_i} - \frac{\partial A_i^n}{\partial \lambda_j}. \tag{4}$$

The Berry phase can alternatively been expressed in terms of the curvature,

$$e^{i\gamma^n} = \exp\left(-i \oint_{\mathcal{S}} \mathcal{F}_{ij}^n d\mathcal{S}^{ij}\right). \tag{5}$$

where **S** is a surface that is bounded by the closed curve **C**, which is readily obtained by applying Stokes' theorem.

2.1 Berry phase for a spin-$1/2$ particle in a magnetic field B

We will apply this formula specifically to calculate the Berry phase for a spin-1/2 particle in a magnetic field, **B**, in three dimensions. The Hamiltonian is given by

$$H = \mathbf{B} \cdot \sigma = B_x\sigma_x + B_y\sigma_y + B_z\sigma_z, \tag{6}$$

where $\sigma_{x,y,z}$ are the 2×2 Pauli matrices. The parameter vector λ in the above expressions for γ^n corresponds in this case to the magnetic field $\mathbf{B} = (B_x, B_y, B_z) = B(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$, where $B = |\mathbf{B}|$.

The orthonormal eigenstates of this Hamiltonian are given by

$$|+B\rangle = \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) e^{-i\phi} \\ \sin\left(\frac{\theta}{2}\right) \end{pmatrix}, \qquad \text{and} \qquad |-B\rangle = \begin{pmatrix} \sin\left(\frac{\theta}{2}\right) e^{-i\phi} \\ -\cos\left(\frac{\theta}{2}\right) \end{pmatrix}, \tag{7}$$

with eigenvalues $+B$ and $-B$, respectively (with them being degenerate when $B = 0$). The Berry connection will be defined over the space of angles, namely θ, ϕ and the magnitude B of the magnetic field.

a) Compute the Berry connection A_θ, A_ϕ, and the Berry curvature $\mathcal{F}_{\theta\phi}$ for both the eigenstates $|\pm B\rangle$.

b) From the expression you get, deduce that the Berry curvature for the state $|+B\rangle$ is given by (in the cartesian coordinate)

$$\mathcal{F}^+_{ij} = \frac{1}{2}\epsilon_{ijk}\frac{B_k}{B^3}, \tag{8}$$

which is equivalent to the field generated by a monopole of strength $1/2$ located at the origin. The location of this monopole corresponds to the point where the two eigenstates $|+B\rangle$ and $|-B\rangle$ become degenerate. Does the interpretation change if we repeat the same calculation for the Berry curvature corresponding to the $|-B\rangle$ eigenstate?

c) Show that the Berry phase corresponding for a closed curve $\mathcal{C}$ in the parameter space that does not include the degenerate point is given by

$$\gamma^+ = -\frac{1}{2}\Omega(\mathcal{C}), \tag{9}$$

where $\Omega(\mathcal{C})$ is the solid angle that the curve $\mathcal{C}$ subtends from the degeneracy point at $\mathbf{B} = 0$. Does the phase depend on which of the two enclosing surfaces we take? **Hint:** Recall Gauss's Law in electrodynamics.

d) We can get what's called a *Chern number* for a closed surface and a Berry curvature by integrating the Berry curvature over this surface–it's quantized in units of 2π such that

$$-\oint_{\mathcal{S}} d\mathcal{S}^{ij}\mathcal{F}_{ij} = 2\pi C, \tag{10}$$

where $C \in \mathbf{Z}$ is the Chern number - an integer. Calculate the Chern number C^+ corresponding to the curvature $\mathcal{F}^+_{ij}$ for the case of a closed surface $\mathcal{S}$ that encloses the degeneracy point.

2.2 Chern insulator

The Berry connection can also be defined on the space of states, as opposed to the space of parameters in the Hamiltonian. One place this is natural is when we have free fermions hopping a lattice with discrete translation symmetry.

We will use this idea to study Chern insulators, which exhibit the quantized Hall response without any external magnetic field (unlike in Integer Quantum Hall effect, where one needs an external magnetic field)! In this problem you will not derive many of the formulae, just use some of the results related to Berry phases (without proof) to numerically discover the Hall response. So, while the actual writeup seems long, the final computation will be rather simple.

We will focus on a two dimensional square lattice, with lattice constants $a(\equiv 1)$ in both directions. Recall, the eigenstates are given by Bloch states,

$$\psi_k(x) = e^{ik\cdot x} u_k(x), \tag{11}$$

where the crystal momenta k are defined over the Brillouin zone, $-\pi/a < k_{x,y} \leq \pi/a$, and $u_k(x+1) = u_k(x)$. Notice, the crystal momenta live on the unit torus T^2. Now, we will investigate the topology of the space of states as we move around the Brillouin zone. We can define a Berry connection, curvature, and Chern number (which is an integer for each band),

$$A_i(k) = -i \langle u_k | \frac{\partial}{\partial k_i} | u_k \rangle , \tag{12}$$

$$\mathcal{F}_{xy} = \frac{\partial A_y}{\partial k_x} - \frac{\partial A_x}{\partial k_y}, \tag{13}$$

$$C = -\frac{1}{2\pi} \int_{T^2} d^2 k \mathcal{F}_{xy}. \tag{14}$$

Next, we will consider the simplest model of a Chern insulator, which is a two band model of a Dirac fermion on two dimensional lattice, with the Bloch Hamiltonian,

$$\tilde{H}(k_x, k_y) = \underbrace{(\sin k_x)\sigma_x + (\sin k_y)\sigma_y + (\cos k_x + \cos k_y + m)\sigma_z}_{\mathbf{E}(k)\cdot\sigma}, \tag{15}$$

Note that this looks a lot like a spin half in magnetic field from the previous problem; except now the parameter space is the space of crystal momenta $k_x, k_y \in T^2$ (with m as an external parameter which we are not sweeping over), as opposed the 3 components of a magnetic field in three dimension. Furthermore, this Bloch Hamiltonian arises naturally on the lattice without any external magnetic field.

You may (should!) use Mathematica for the computations that follow.

a) Compute the two eigenstates and eigenvalues of the Bloch Hamiltonian as a function of k_x, k_y, m. Observe that the spectrum becomes gapless at certain momenta at $m = 0, -2, +2$.

b) Away from the gapless points at $m \in \{-2, 0, +2\}$, the system can be made to be an insulator by tuning the chemical potential in the gap and filling the lower band.

   We want to compute the Chern number of this band. In fact, it is useful to introduce a mapping between the 2 torus to a 2 sphere $T^2 \to S^2$, by introducing a unit three vector,

$$\mathbf{n}(k) = \frac{\mathbf{E}(k)}{|\mathbf{E}(k)|}. \tag{16}$$

   One can show (you don't have to prove, but you are welcome to try!), that under this mapping, the formula for the Chern number becomes,

$$C = \frac{1}{4\pi} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} d^2k \;\; \mathbf{n} \cdot \left(\frac{\partial \mathbf{n}}{\partial k_x} \times \frac{\partial \mathbf{n}}{\partial k_y} \right). \tag{17}$$

   Compute the Chern number numerically in the gapped regions $m < -2, -2 < m < 0, 0 < m < 2, m > 2$.

c) Finally, one can relate the Hall conductance to the Chern number of the band by the famous TKNN (Thouless, Kohomoto, Nightingale and den Nijs) formula,

$$\sigma_{xy} = \frac{e^2}{2\pi\hbar} \sum_{a \in \text{filled bands}} C_a. \tag{18}$$

Argue that from your previous result you get that the Hall conductivity in the Chern insulator is quantized to non-zero values in different gapped regions.