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Tutorial 3: Toric code

1 The Toric Code Model

In this tutorial we will review the basics of the Toric Code model [1].

Consider a square lattice with one spin 1/2 degree of freedom (basis states $|0\rangle$ and $|1\rangle$) per each edge.

The Hamiltonian contains two types of terms: one involving four τ^x's around a vertex v, one involving four τ^z's around a plaquette p.

$$H = -\sum_{v}\Big(\prod_{v\in e}\tau_e^x\Big) - \sum_{p}\Big(\prod_{e\in p}\tau_e^z\Big) \tag{1}$$

a) Check that all the terms in the Hamiltonian commute with each other. How can we describe the ground state in terms of the local stabilizers?

b) We will find a nice pictorial way to think of the wave-function. We can think of the spin 1/2 degrees of freedom as a Z_2 string on each edge. That is, the $|+\rangle$ state corresponds to no string on each edge while the $|-\rangle$ state corresponds to the existence of a string on the edge. Consider states that minimize the the energy of the $-\prod_{v\in e}\tau_e^x$ terms for each vertex. What are the allowed local configurations of the strings around the vertex? What does this say about the allowed global configurations of the strings?

c) Now consider the action of the plaquette terms $\prod_{e\in p}\tau_e^z$ on the subspace of allowed string states. What does the ground state wavefunction look like in this basis?

d) Now let us pay more attention to the boundary conditions. Consider the system on a torus. Can you argue what the ground state degeneracy is from the pictorial description of the ground state wavefunction that you get?

[1] The tutorial is adapted from notes by Xie Chen `https://xiechen.caltech.edu/documents/27793/UQM_lecture.pdf`

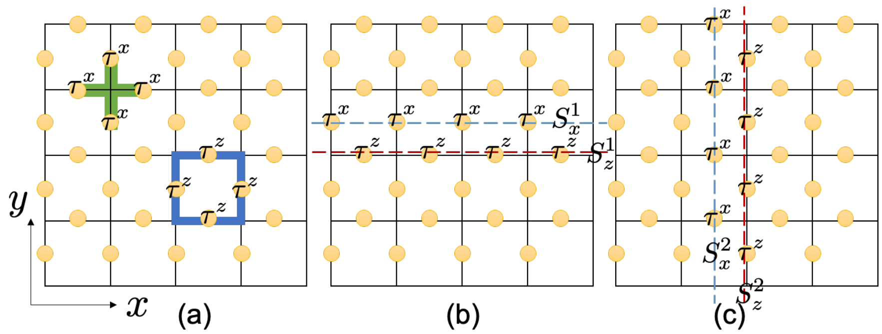

Figure 1: The Toric Code model: (a) Hamiltonian terms, (b) (c) logical operators along nontrivial loops.

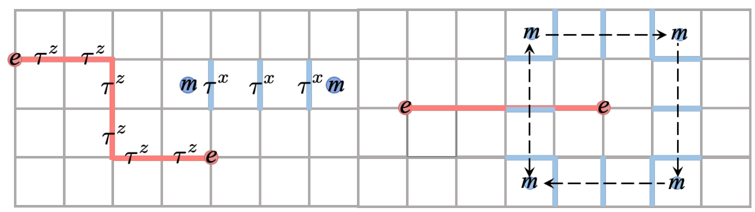

Figure 2: Creation of fractional excitations in the Toric Code model and their braiding.

e) Now we will look for operators that can distinguish between the different ground states. Consider the $S^z = \prod \tau^z$ string operators along the nontrivial loops in the x and y direction, and the $S^x = \prod \tau^x$ string operators along the nontrivial loops on the dual lattice. Argue that these operators commute with the Hamiltonian but anticommute with each other in a specific way. Explain how this describes the groundstate degeneracy, and can be interpreted as logical operators in vieiwing the toric code as a quantum error correcting code.

f) When the string operator ends, Hamiltonian terms are violated and anyonic excitations are made in pairs (at each ends), as shown in Fig. 2. The ends of the S^z operator are the electric e excitations; the ends of the S^x operator are the magnetic m excitations. We will compute the exchange and braiding statistics according to the figure 3.

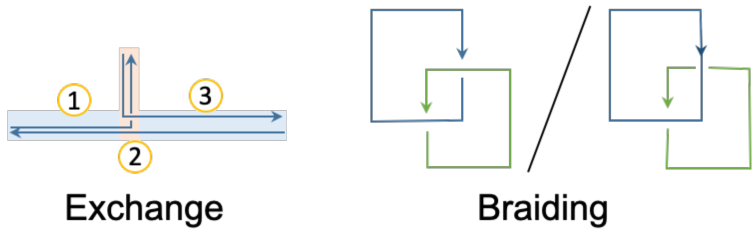

Figure 3: String operator configurations for calculating the exchange and braiding statistics of fractional excitations. The exchange statistics can be calculated in the three steps shown. The string operators in the same shaded areas overlap with each other. They are drawn as apart only for clarity.

Show that the e and m anyons are self-bosons and mutual fermions!

2 TC from Gauging the Ising Paramagnet

We will see in the lectures that the Toric code "emerges" a Z_2 gauge theory on the lattice. For more details on why exactly the toric code is a gauge theory, see Sec. 3 of Xie Chen's notes, or John McGreevy's notes on Quantum Phases of Matter (Spring 2024). In this problem, we will just follow some computations that build more intuition about the emergence of toric code by gauging the Ising paramagnet.

It is useful to just use the following dictionary for interpreting the energy terms of the toric code Hamiltonian as terms in Z_2 gauge theory:

$$\prod_{v \in e} \tau_e^x \to e^{i\pi \sum_{v \in e} E_e}, \quad \prod_{e \in p} \tau_e^z \to e^{i \sum_{e \in p} A_e} \tag{2}$$

,

where E and A are Z_2 valued equivalents of the electric field and the electromagnetic potential respectively.

Therefore, imposing these two terms correspond to imposing the gauge symmetry (Gauss's law) around vertex v and imposing the zero flux condition around plaquette p respectively.

As a Z_2 gauge theory, Toric code can be obtained by 'gauging' a model with global Z_2 symmetry. That is, to promote the global Z_2 symmetry in the model to a local Z_2 symmetry.

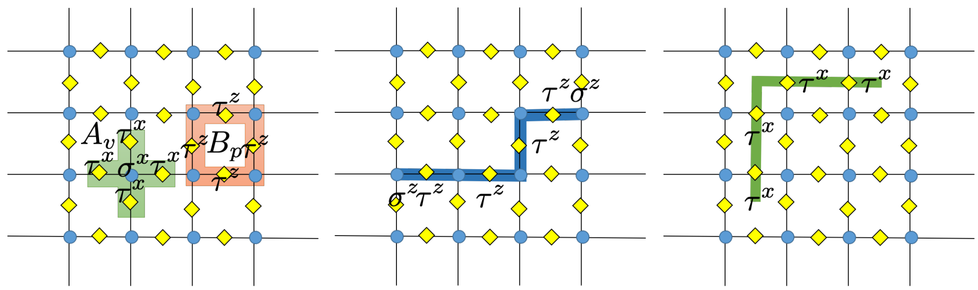

Figure 4: Gauging the Z2 symmetry in the transverse field Ising model to obtain the Toric Code model.

In particular, it can be obtained by 'gauging' the trivial paramagnet in the transverse field Ising model. Let's see how that is achieved.

In the transverse field Ising model, the Hamiltonian at the paramagnetic limit (no spontaneous symmetry breaking) takes the simple form of

$$H = -\sum_v \sigma_v^x \tag{3}$$

where σ^x acts on the spin 1/2 degrees of freedom on each lattice site (circles in Fig.4) as $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. This is the 'matter' DOF of the system. The system has a global Z_2 symmetry of $U = \prod_v \sigma_v^x$ and the ground state is invariant under this symmetry

$$|\psi\rangle = \otimes \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle) \tag{4}$$

where $|\uparrow\rangle$ and $|\downarrow\rangle$ are eigenstates of $\sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Our goal is to take the system with global symmetry and turn it into a system with local symmetry. That is, the symmetry group will be generated not by just one global symmetry operation, but by symmetry operations at all spatial locations (on each lattice site). Note that, the transverse field Hamiltonian is already invariant under local Z_2 symmetries generated by σ^x on each site. But this is no longer true if a small Ising perturbation of the form $\sigma^z\sigma^z$ is added. We want the procedure to work for the whole phase. Therefore, although we will focus on the transverse field limit for simplicity of discussion, we will present a gauging procedure that can be applied to any Z_2 symmetric Hamiltonian.

The gauging procedure can be generalized to all kinds of global internal unitary symmetries of group G in the following steps:

- Take a system with global unitary internal symmetry of group G with degrees of freedom on the sites of a lattice and with global symmetry acting as a tensor product of operators on each lattice site.
- Introduce gauge field degrees of freedom onto the edges of the lattice and define gauge symmetry transformation as acting on each lattice site and the edges around it.
- Modify the terms in the original Hamiltonian, by coupling the original degrees of freedom with the gauge field, such that each term is invariant under all local gauge symmetry transformations.
- Add vertex terms to the Hamiltonian to enforce gauge symmetry and add plaquette terms to enforce the zero flux constraint.

To do this, first we introduce the gauge field degrees of freedom on each edge of the lattice (diamonds in Fig.4). For a Z_2 gauge field, the degrees of freedom are again two level spin 1/2's and we label them as τ spins. We define a local Z_2 symmetry generated at each lattice site with $U_v = \sigma_v^x \prod_{v\in e} \tau_e^x$ where the product is over all edges with v as one end point. This local symmetry can be interpreted as enforcing the Gauss's law in the presence of charged matter.

a) Now, follow the steps mentioned above, interpreting the zero flux constraint as adding a term in the Hamiltonian as $-\sum_p \prod_{e\in p} \tau_e^z$ (for an explanation why, read the next section after the computations). What is the Hamiltonian that you get after you do the gauging? Argue, why it is the following:

$$H_g = -\sum_i \sigma_i^x - \sum_v U_v - \sum_p F_p = -\sum_i \sigma_i^x - \sum_v \sigma_v^x \prod_{v\in e} \tau_e^x - \sum_p \prod_{e\in p} \tau_e^z. \tag{5}$$

b) Now argue that by enforcing the gauge symmetry as a constraint, we get exactly the toric code!

c) The usefulness of the interpretation of the toric code as a gauge theory comes from the fact that its properties can be interpreted as a property of the entire phase in a gauge theory, and not just for the isolated toric code Hamiltonian.

To see this, we will now see that the mutual and self statistics of the (anyonic) quasi-particles are still true in the above Hamiltonian when the U_v is not enforced as a hard constraint and the matter DOF are kept in the system.

Check now that the string operators can take the form shown in Fig. 4 (b), (c). The one of the form $\sigma^z \tau^z \tau^z ... \tau^z \sigma^z$ creates a gauge charge e at its two ends, in a way that does not violate the gauge constraints. The one of the form $\tau^x \tau^x ... \tau^x$ creates a π gauge flux m at its two ends. Using the procedure in Fig. 3, show that both the gauge charge and gauge flux are bosons and they have a -1 mutual braiding statistics.