Subhayan Sahu, Leo Lessa, Chong Wang

Tutorial 4: More Toric code, Landau levels

1 Long range entanglement in Toric code groundstate

In the lecture, we saw that local indistinguishability in toric code groundstate implies that any state $|\psi\rangle$ in the ground state subspace is long-range entangled, i.e. there can not be any finite depth unitary circuit U_{FD} that connects it to a trivial product state, i.e., such that $|\psi\rangle \neq U_{FD}\,|+ + \cdots +\rangle$. We will see this more explicitly in the first problem of the tutorial, and also show a different proof of this above fact without using local indistinguishability.

a) Let us first recall the proof of long range entanglement from local indistinguishability. Assume that $|\psi\rangle$ is connected to a product state $|+ + \cdots +\rangle$ via finite-depth quantum circuit U_{FD}, as $|\psi\rangle = U_{FD}\,|+ + \cdots +\rangle$, and that another toric code groundstate $|\phi\rangle$, orthogonal and locally indistinguishable to $|\psi\rangle$, exists to reach a contradiction.

   Hint: The only state locally indistinguishable to $|+ + \cdots +\rangle$ *is itself.*

b) Does this argument work if you want to prove that the toric code ground state on a sphere (instead of a torus) is long range entangled?

c) We would like to argue that any topologically ordered ground state hosting anyons with non-trivial braiding statistics must be long range entangled, without resorting to the global topology of the base manifold.

   Consider an operator string γ_m^{ab} that creates two $m-$particles (plaquette excitations) along a path γ^{ab} connecting two (plaquette) sites a and b through the upper half-plane R^{up}. Next, consider an operator loop $\gamma_e^{\circ_b}$ that moves an e particle (a vertex excitation) along a closed loop $\circ_b$ around the point b, and its inverse $(\gamma_e^{\circ_b})^\dagger$ (See Fig. 1). If $|\psi\rangle$ is a toric code ground state, what is the effect of the braiding of these operators on such state, i.e. what is the final state $(\gamma_e^{\circ_b})^\dagger\gamma_m^{ab}\gamma_e^{\circ_b}\,|\psi\rangle$?

d) By contradiction, let us assume that $|\psi\rangle$ is short-range entangled. By definition, this means there exists a finite-depth unitary circuit U_{FD} such that $|\psi\rangle = U_{FD}\,|+ + \cdots +\rangle$.

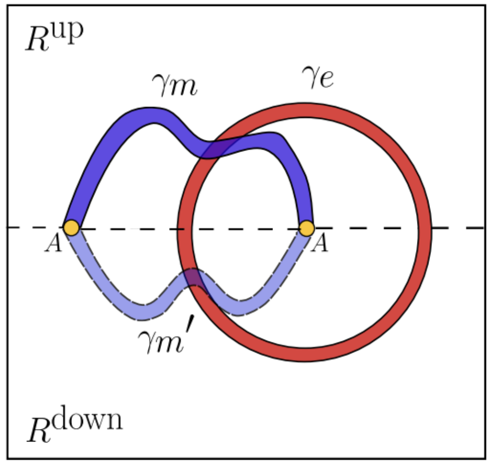


Figure 1: Anyon braiding diagram describing the action of the three operators. Here the two dots are the points a, b which constitute a subregion A. We further divide the rest of the 2d surface into R^{up} and R^{down} which are mutually exclusive subregions, such that $A \cup R^{up} \cup R^{down}$ is the whole lattice (figure from Lee, Li, Yoshida `https://arxiv.org/pdf/2405.07970`.)

The loop operator $\gamma_e^{\circ_b}$ is a symmetry of the toric code ground state $|\psi\rangle$, since $\gamma_e^{\circ_b}|\psi\rangle = |\psi\rangle$. Prove that the same is true for $|+ + \cdots +\rangle$ under the "dressed" loop operator $\tau_e^{\circ_b} \equiv U_{FD}^\dagger \gamma_e^{\circ_b} U_{FD}$.

e) Consider another m-particle string operator $\gamma_m'^{ab}$ connecting a and b though the lower half-plane R^{down}. Together, γ_m^{ab} and $\gamma_m'^{ab}$ form a loop operator via $\gamma_m^\circ \equiv \gamma_m^{ab}(\gamma_m'^{ab})^\dagger$.

Even though only the loop operator γ_m° is a symmetry of $|\psi\rangle$, and not the strings γ_m^{ab} and $\gamma_m'^{ab}$ (why?), the same does not happen with a product state such as $|+ + \cdots +\rangle$. In fact, we can split the dressed counterpart τ_m° into two open-string operators that are also symmetries of $|+ + \cdots +\rangle$. The one acting acting non-trivially on the upper half-plane is defined as

$$\tilde{\tau}_m^{ab} \equiv {}_{R^{\mathrm{down}}}\langle + + \cdots + | \, \tau_m^\circ \, | + + \cdots + \rangle_{R^{\mathrm{down}}} \otimes \mathbb{1}_{R^{\mathrm{down}}}. \tag{1}$$

Argue that

i) $\tilde{\tau}_m^{ab}$ is a symmetry of $|+ + \cdots +\rangle$;

ii) $\tilde{\tau}_m^{ab}$ acts in the same way as $\tau_m^{ab} \equiv U_{FD}^\dagger \gamma_m^{ab} U_{FD}$ far from its endpoints. That is, $\tilde{\tau}_m^{ab} = \tau_m^{ab} O_A$, where A is a small region around a and b (See Fig. 1) and O_A is an operator supported in A; and

iii) the braiding relation is preserved: $(\tau_e^{\circ_b})^\dagger \tilde{\tau}_m^{ab} \tau_e^{\circ_b} = -\tilde{\tau}_m^{ab}$. Here, we assume that the curves γ_m and γ_e intersect at a point far away from A.

f) Finally, use part d) and points i) and iii) above to reach a contradiction.

g) In words, why did we reach a contradiction? Where was the fact that $|+ + \cdots +\rangle$ is a product state and that U_{FD} has finite depth important?

2 Toric code with boundaries (topological qubit)

Consider the toric code on a lattice with boundaries as shown in Fig 2,

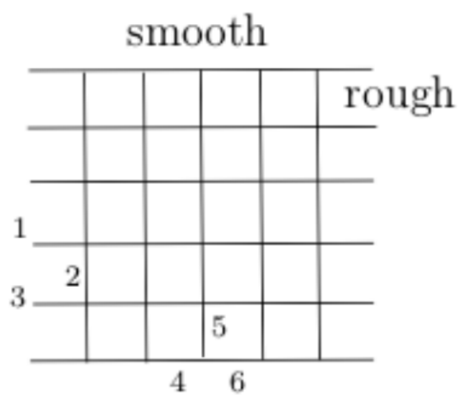
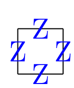
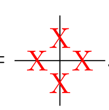


Figure 2: Toric code on boundaries

The rough boundary conditions means that plaquette terms get truncated, such as the term $-Z_1Z_2Z_3$, while smooth boundary conditions mean that star terms get truncated, such as the term $-X_4X_5X_6$. Here, the plaquette B_p and the star A_v terms are defined in the bulk with Z and X operators respectively:

$B_p =$ 

, $\quad A_v =$ 

. (2)

Show that there is a two-dimensional space of groundstates. A good way to do this is using the algebra of string operators which terminate at the various components of the boundary without creating excitations.

3 Landau levels

In this problem we will consider a charged particle in a uniform magnetic field $\mathbf{B} = B\hat{z}$. We will ignore the dimension in which the field is pointed, so the particle moves only in the two directions x, y transverse to the field. This problem is a crucial ingredient in the quantum Hall effect(s), which will be a topic of study in this week and the next. This problem is for you to jog your memory before the lectures.

Consider a particle of charge q in a vector potential

$$\mathbf{A} = \frac{B}{2}(-y\hat{x} + x\hat{y}). \tag{3}$$

a) Show that the magnetic field is as stated above.

b) Show that a classical particle in this potential will move in circles at an angular frequency $\omega_0 = \frac{qB}{mc}$ where m is the mass and speed of light $c = 1$.

c) Consider the Hamiltonian for the corresponding quantum problem

$$\begin{aligned} H &= \frac{1}{2m}|\mathbf{p} - q\mathbf{A}|^2 &(4) \\ &= \frac{1}{2m}\left(\left(p_x + \frac{qB}{2}y\right)^2 + \left(p_y - \frac{qB}{2}x\right)^2\right). &(5) \end{aligned}$$

Show that

$$Q \equiv \left(p_x + \frac{qB}{2}y\right) \quad \text{and} \quad P \equiv \left(p_y - \frac{qB}{2}x\right) \tag{6}$$

are canonical in the sense that $[Q, P] = i\hbar qB$. Write H in terms of these operators and show that the allowed levels are $E_n = \left(n + \frac{1}{2}\right)\hbar\omega$. What is ω? It is convenient to construct the creation and annihilation operators $a = \frac{1}{\sqrt{2q\hbar B}}(Q + iP)$, $a^\dagger = \frac{1}{\sqrt{2q\hbar B}}(Q - iP)$ and checking that $[a, a^\dagger] = 1$.

d) Can you argue (in this gauge) why each energy level must be degenerate? Hint: find another canonical pair of operators that commutes with H and with Q, P.

e) To understand the degeneracy better, let's write the wavefunctions for $n = 0$ (the lowest Landau level (LLL)) in terms of $z \equiv x + iy$, $\overline{z} \equiv x - iy$. Recall that the groundstate(s) of a harmonic oscillator satisfy $a|0\rangle = 0$. Write this condition for the $n = 0$ states in terms of $z, \overline{z}$. Writing the LLL wavefunctions as

$$\psi_0(z, \overline{z}) = \langle x, y|n = 0\rangle = e^{-\frac{qB}{4\hbar}|z|^2} u(z, \overline{z}) \tag{7}$$

show that the condition is solved when $u(z, \overline{z})$ is any holomorphic function:

$$\partial_{\overline{z}} u = 0. \tag{8}$$

f) A useful basis of such functions is monomials $u_m = z^m$. Show that $\psi_{0,m} \equiv z^m e^{-\frac{1}{4\ell_B^2} zz^*}$ (where $\ell_B \equiv \sqrt{\frac{\hbar c}{qB}}$ is the magnetic length) is peaked at a radius $r_m = \sqrt{2m}\ell_B$.

g) Show that $\psi_{0,m}$ is an eigenstate of the angular momentum $L_z = i(xp_y - yp_x) = i\hbar\partial_\phi$, where $z \equiv re^{i\phi}$.

h) If the system is a disc of radius R there is a biggest value of m that can fit. Show that the number of LLL states that can fit is

$$N = \frac{\Phi_B}{\Phi_0} \tag{9}$$

where $\Phi_B = \pi R^2 B$ is the flux through the sample and $\Phi_0 \equiv \frac{2\pi\hbar c}{q}$ is the flux quantum which appears in the periodicity of the interference pattern in the Aharonov-Bohm experiment.

.