Subhayan Sahu, Leo Lessa, Chong Wang

Tutorial 5: BKT and Edge states

1 2D XY model

[1]

The 2D XY model describe superfluid He-4 and He-3 films as well as thin superconducting films. In the lecture we saw that the ground state of a system with $U(1)$ symmetry can describe an insulator if the $U(1)$ symmetry is preserved, and a superfluid if the $U(1)$ symmetry is spontaneously broken. We also saw that there are non-trivial topological excitations called vortices on top of the superfluid ground state which can bind magnetic fluxes. However, in the absence of external electromagnetic field, the vortices do not appear in the ground state as they are energetically unfavorable. The discussion in the lectures strictly considered the zero temperature regime, where it was sufficient to consider the ground state. In this problem we will consider what happens at finite temperature for 2D $U(1)$ symmetric models, starting from the spontaneous symmetry broken phase.

Neglecting fluctuations in the amplitude of the order parameter, the simplest form of the potential energy dependence on the phase degree of freedom $\varphi(r)$ (the angle coordinate in the $U(1)$ degree of freedom) consistent with the $U(1)$ symmetry is

$$U = \frac{1}{2}\rho_s \int d^2 r |\nabla \varphi|^2, \tag{1}$$

where ρ_s is called the (bare) 'spin stiffness' or 'superfluid density' and in 2D has units of energy. It sets the characteristic temperature scale for the system.

a) Consider the partition function at inverse temperature β

$$Z = \int D\varphi \, e^{-\beta \frac{1}{2}\rho_s \int d^2 r |\nabla \varphi|^2}, \tag{2}$$

[1] This problem is adapted from Steven Girvin's notes on the BKT transition: https://boulderschool.yale.edu/sites/default/files/files/kosterlitz-thouless.pdf.

where $\int D\varphi$ indicates a functional integral or partition sum over all possible configurations of the field φ. Does this system have a symmetry breaking phase transition at any finite temperature? How is your answer consistent with what happens at zero temperature? *Hint: Count the effective spacetime dimension and pay attention to the nature of the symmetry.*

b) Now, let us explore then what the correlation function looks like for the above Gaussian model. Use that fact that for a gaussian model

$$G(r) \equiv \langle e^{-i\varphi(r)} e^{i\varphi(0)} \rangle = e^{-\frac{1}{2}\langle [\varphi(r)-\varphi(0)]^2 \rangle}. \tag{3}$$

Also using the fact that the quadratic form is diagonal in Fourier space we have,

$$\langle \varphi_{k_1} \varphi_{k_2} \rangle = \frac{\delta_{k_1,-k_2}}{\beta \rho_s k_2^2}, \quad \varphi_k = \frac{1}{(2\pi)^2} \int d^2 k e^{ik\cdot r} \varphi_r. \tag{4}$$

Use these formula to estimate the function $G(r)$ for $r \gg a$, where a is a lattice spacing that provides an UV cutoff for $k < 1/a$. You will find that the integral expression also diverges in the infra-red, which can be cur off by $1/r$. Does the correlation function $G(r)$ have long range order ($G(r) \neq 0$ at $r \to \infty$), exponential decay $G(r) \sim e^{-r}$, or some other behavior? Is this consistent with your answer from the previous section?

c) Does this answer for $G(r)$ make sense at $T \gg \rho_s$? We expect any long range order to give in to exponential decay at high enough temperatures.

d) Our gaussian model has neglected the existence of topological defects, vortices, in the order parameter (see examples in Fig. 1).

To incorporate the effects of the vortices, we introduce a lattice regularization

$$H = -J \sum_{\langle ij \rangle} \cos(\varphi_i - \varphi_j), \tag{5}$$

where i and j label lattice sites on (say) a square lattice of lattice constant a and the sum is over near neighbors.

Defining 2D spin vectors $\mathbf{S}$ by $S_x + iS_y = e^{i\varphi}$, we can map this onto a model of a 2D magnet with easy plane anisotropy

$$H = -J \sum_{\langle ij \rangle} \mathbf{S}_i \cdot \mathbf{S}_j. \tag{6}$$

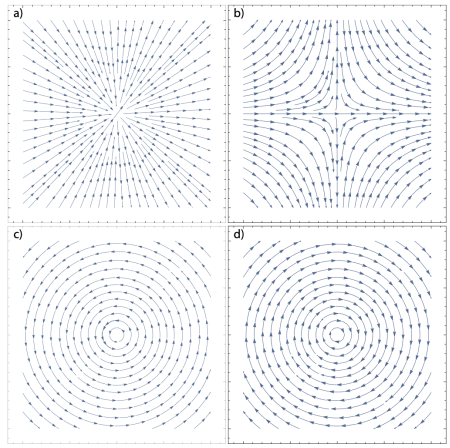

Fig. 1. Examples of systems with topological defects. a) Streamlines for the vortex θ_+. b) Streamlines for the antivortex θ_-. c) The vortex's gradient, $\nabla\theta_+$. d) The antvortex's gradient, $\nabla\theta_-$. Circulation quantifies how the vectors in c) and d) rotate when integrating along a closed path.

Figure 1: Fig from `https://phas.ubc.ca/~berciu/TEACHING/PHYS502/PROJECTS/18BKT.pdf`

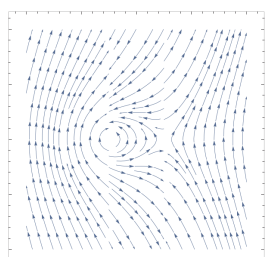

Fig. 2. A rough plot of a vortex-antivortex pair embedded in an ordered system, in which all spins were originally pointing upwards

Figure 2: Fig from `https://phas.ubc.ca/~berciu/TEACHING/PHYS502/PROJECTS/18BKT.pdf`

If we assume $T \ll J$ so that the spins are nearly parallel on neighboring sites then we can expand the cosine to second order to obtain the lattice gaussian model

$$H \approx \frac{1}{2} J \sum_{\langle ij \rangle} (\varphi_i - \varphi_j)^2. \tag{7}$$

The continuum approximation to this model is identical to the gaussian model if we take

$$J = \rho_s. \tag{8}$$

This is the reason that ρ_s is often referred to as the 'spin stiffness' instead of the superfluid density.

The Gaussian model has the usual global symmetry under a shift in the angle $\varphi_r \to \varphi_r + \delta\theta$. However, the lattice regulation is also invariant under discrete local transformations which change any single spin. Identify this extra symmetry and argue why that allows for vortices.

e) A vortex is a topological defect in which the phase winds by $\pm 2\pi$ in going around the defect as illustrated in Figs. (1-2)

$$\oint dr \cdot \nabla\varphi = 2\pi n_W \tag{9}$$

where $n_W = \pm 1$ is the topological 'charge' or winding number. In principle it is possible to have vortices with higher winding numbers but these are generally expensive energetically and can be safely ignored. In the presence of a vortex there is a discontinuity in φ where the 2π value is adjacent to the $\varphi = 0$ value. What is the winding number of the vortex and antivortex (fig 1) and the vortex-antivortex pair (fig2)?

f) We can now ask ourselves how much energy it costs to introduce a vortex into the system. In the continuum limit the phase field configuration for a right-handed vortex centered on the origin as shown in Fig. (1) is simply

$$\varphi(r) = \theta(r) + \theta_0 \tag{10}$$

where $\theta = \arctan(y/x)$ is the azimuthal angle at position r and θ_0 is an arbitrary constant. Hence we have $\nabla\varphi = \hat{\theta}/r$. Estimate the energy cost in a system of size L.

g) The integration at large distances diverges logarithmically with system size. We thus see that the vortex costs an infinite amount of energy in the thermodynamics limit. However they are still relevant at finite temperatures, as can be seen by computing the entropy. Estimate the entropy and the free energy of the vortices. Do you see a thermodynamic transition as you tune temperature?

h) This thermal phase transition is the famous Berezinskii–Kosterlitz–Thouless transition. Explain physically what happens in terms of the vortices above and below the transition, and how that affects the correlation functions.

2 Edge states for integer quantum Hall effect

In the last tutorial, we studied Landau levels. In this problem we will highlight their edge states, which are very important for understanding the quantum Hall phenomena.

a) First let us do a very rough computation that recaps the known Landau level phenomena. We know what classical electrons in a perpendicular magnetic field go around in cyclotron orbits, because of the Lorentz force. The cyclotron radius in a magnetic field B for an electron with velocity v is $r_c = mv/eB$. An electron in a cyclotron orbit at velocity v has angular momentum $L = mvr_c$. When quantized, only orbits with a quantized angular momentum $L = n\hbar$ will be allowed. Argue from these observations that only discrete values of radius are allowed, and that the energy spectrum looks like that of harmonic oscillator. Recall that each quantized level of the Harmonic oscillator is highly degenerate: one electron can be added in each level for every flux quantum of the magnetic flux passing through the system. Therefore Landau levels have a huge degeneracy, proportional to the area of the sample.

b) Next, we will argue that something special happens along the edge of a quantum Hall system, which can be seen even classically. For a fixed magnetic field and a charge of the particle, all particle motion is in one direction, say anti-clockwise. What happens to this simple picture at the edge of the sample? A particle restricted to move in a single direction along a line is said to be chiral. Argue that this implies there will be chiral edge modes in the quantum Hall sample.

c) Next, we will make a more careful quantum argument. The edge of the sample is modeled by a potential which rises steeply as shown in the figure below. We'll work in Landau gauge and consider a rectangular geometry which is finite only in the x-direction, which we model by $V(x)$. The Hamiltonian in this gauge is

$$H = \frac{1}{2m}\left(p_x^2 + (p_y + eBx)^2\right) + V(x) \tag{11}$$

In the absence of the potential, we know that the wavefunctions are Gaussian of width l_B. If the potential is smooth over distance scales l_B, then, for each state, we can Taylor

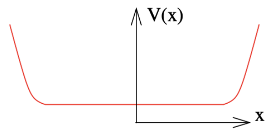

Figure 3: Figure from David Tong's lectureshttps://www.damtp.cam.ac.uk/user/tong/qhe/qhe.pdf

expand the potential around its center's location X. Each wavefunction then experiences the potential $V(x) \approx V(X) + \frac{\partial V}{\partial x}(x - X) + \cdots$. Dropping the quadratic terms and the constant term, argue that this looks like a charged particle in a perpendicular magnetic field and a position dependent electric field.

d) Now we will study how the Landau levels are affected by the electric field $E(x) =$. See that the Hamiltonian is now that of a displace Harmonic oscillator, whose energy levels are given by (argue why!),

$$E_{n,k} = \hbar\omega_c(n + 1/2) - eE(x)\left(kl_B^2 + \frac{e\frac{\partial V}{\partial x}}{m\omega_c^2}\right) + \frac{m\left(\frac{\partial V}{\partial x}\right)^2}{2B^2}. \tag{12}$$

Because the energy now depends on the momentum, argue that it means that states now drift in the y direction. What is the drift velocity?

e) Argue that this implies that the modes at each edge are both chiral, traveling in opposite directions. Does this agree with the classical picture of skipping orbits?

Having a chiral mode is violated by a theorem which says that one can't have charged chiral particles moving along a wire; there has to be particles which can move in the opposite direction as well. The reason that the simple example of a particle in a magnetic field avoids this theorem is because the chiral fermions live on the boundary of a two-dimensional system, rather than in a one-dimensional wire.