PERIMETER SCHOLARS INTERNATIONAL

Tensor Networks
Summer 2025
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Module 1

1 Introduction to tensor network diagrammatics

Motivation:

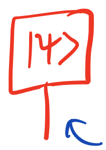
- What is the memory cost to represent a N qubit quantum state?

- What is the memory cost to represent the joint probability distribution of N bits?

Tensors : generalization of vectors and matrices.

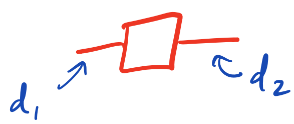
- A d-dimensional vector $|\psi\rangle \in \mathbb{C}^d$; $|\psi\rangle \equiv \sum_{i=1}^{d} \psi_i |i\rangle$

  $\equiv$ A rank-1 tensor / array

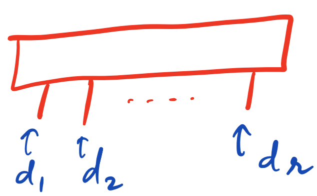


"physical dimension" $= d$

- $d_1 \times d_2$ dimensional matrix $\equiv$ rank-2 tensor $\in \mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$



- $d_1 \times d_2 \times \cdots \times d_n$ dimensional tensor $\in \mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} \cdots \otimes \mathbb{C}^{d_n}$



can be in general specified by $d_1 \times d_2 \times \cdots \times d_n$ complex numbers.

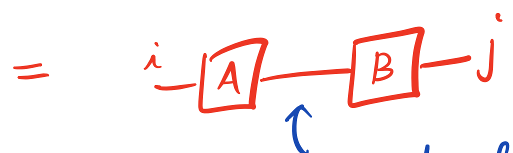
2 Tensor operations

- <u>Matrix products</u>.

$A \to m_1 \times m_2$ dimensional matrix

$B \to m_2 \times m_3$ " "

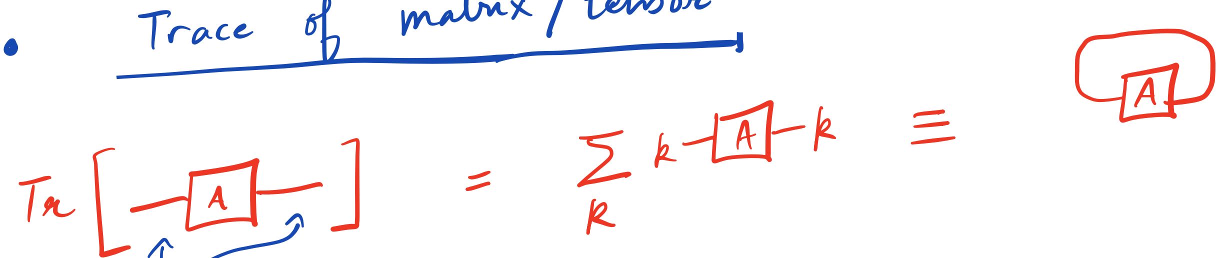
$AB = C \to m_1 \times m_3$ dimensional matrix



contracted legs

- computational complexity
  $= m_2$ elementary steps.

- <u>Trace of matrix / tensor</u>

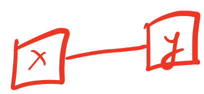


dimensions must match.

Exercise 1. Generalize this to tensors of general rank

2. Diagrammatically prove $\mathrm{tr}(AB) = \mathrm{tr}(BA)$

- **General contractions**

$\langle \vec{x}, \vec{y} \rangle \equiv$ 

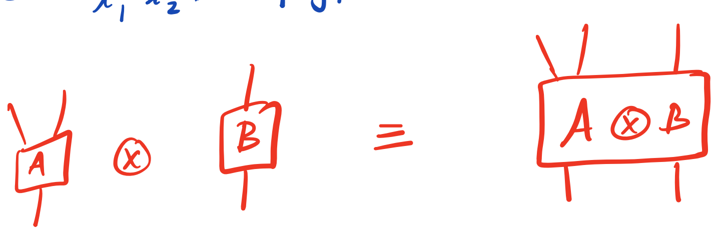
$M\vec{v} \equiv$ 

$AB \equiv$ 

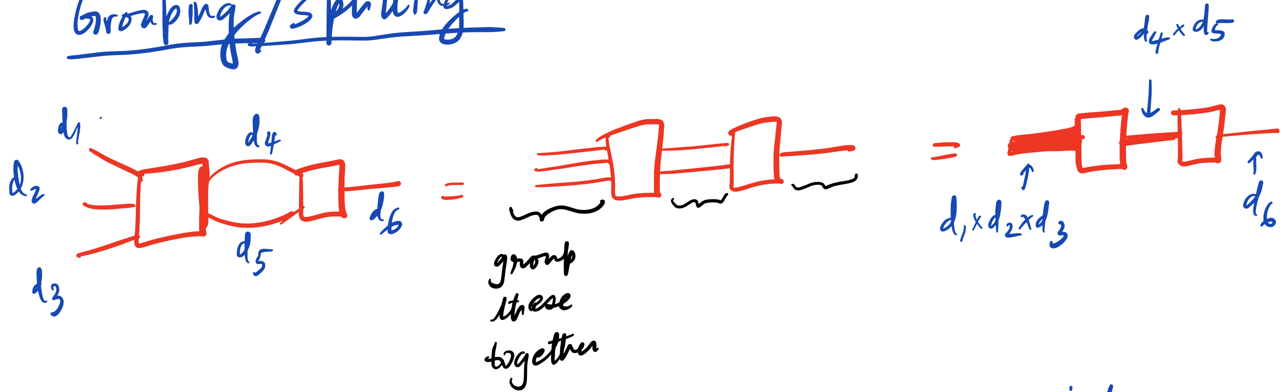
- **Tensor product**

Let A and B be tensors of rank r_1 and r_2.
$A \otimes B$ is a tensor of rank $r_1 + r_2$, defined as

$$[A \otimes B]_{i_1 i_2 \dots i_{r_1} j_1 j_2 \dots j_{r_2}} \equiv A_{i_1 i_2 \dots i_{r_1}} B_{j_1 j_2 \dots j_{r_2}}$$



- ## Grouping/splitting



A rank $n+m$ tensor can be converted into a rank 2 tensor by grouping as such:

$$T_{I,J} := T_{i_1 \dots i_n ; j_1 \dots j_m}$$

$$I := i_1 + d_1^{(i)} \cdot i_2 + d_1^{(i)} d_2^{(i)} \cdot i_3 + \dots + d_1^{(i)} \dots d_{n-1}^{(i)} \cdot i_n$$

$$J := j_1 + d_1^{(j)} \cdot j_2 + d_1^{(j)} d_2^{(j)} \cdot j_3 + \dots + d_1^{(j)} \dots d_{n-1}^{(j)} \cdot j_n$$

The reverse process is "splitting"

3 Splitting tensors and Singular Value Decomposition

SVD of matrix

$$A = U \, S \, V^{\dagger}$$

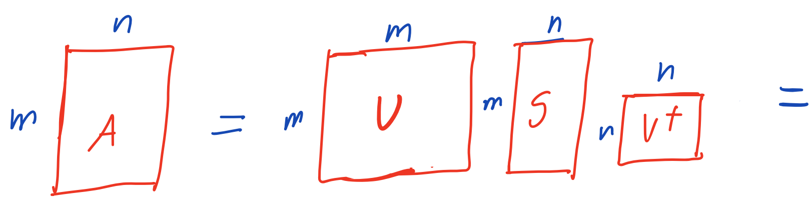
A: $m \times n$

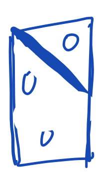
U: $m \times m$ unitary matrix

S: $m \times n$ rectangular diagonal matrix } entries are non-negative real.

$V^{\dagger}$: $n \times n$ unitary matrix

$$U U^{\dagger} = U^{\dagger} U = \mathbb{1}_{m \times m} \qquad V V^{\dagger} = V^{\dagger} V = \mathbb{1}_{n \times n}$$



$S \equiv$ 

→ The entries are the singular values of A.

$$A v = \sigma u$$
$$A^{\dagger} u = \sigma v$$

Columns of U and V form an orthonormal basis

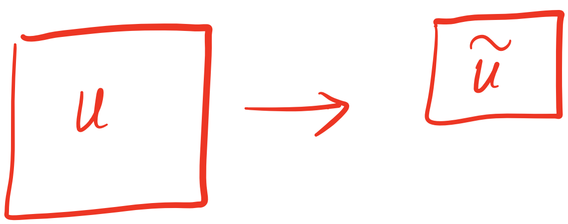
\# of singular values $\equiv$ "rank" of A.

$$\leq \min(m, n)$$

Why SVD ? $\Rightarrow$ Best compression of a matrix.

(i) Because we can "truncate" matrices to a low rank matrix approximaton by keeping only the first k singular values of S, and first k columns of U and V:

$$\tilde{A}^{(k)} \equiv \tilde{U}\,\tilde{S}^{(k)}\tilde{V}^{t}$$ [This has rank k]



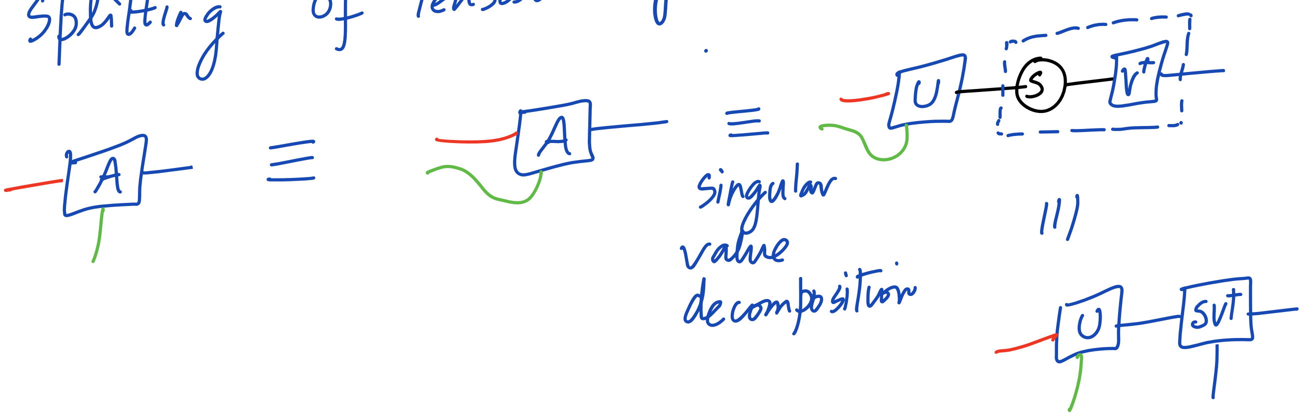
Eckart-Young theorem says $\tilde{A}^{(k)}$ is the optimal matrix closest to A with fixed rank k,

i.e $\|A-\tilde{A}^{(k)}\|_F \le \|A-B\|_F$ for all B of rank k.

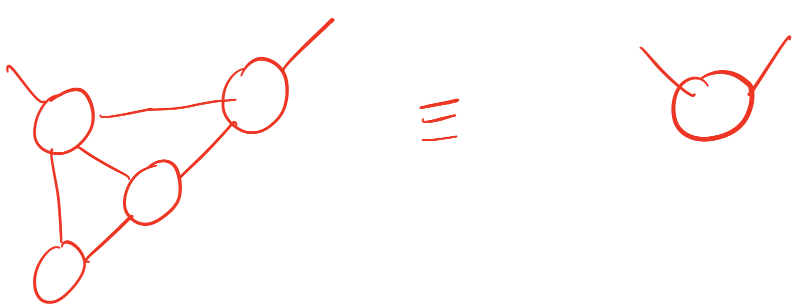
Here $\|M\|_F \equiv \sqrt{\operatorname{tr} M^{\dagger}M}$ is the Frobenius norm.

(ii) Also provides an optimal truncation of entanglement (will be shown later).

Splitting of tensor by SVD :



Tensor Networks : diagram which tells how to combine several tensors into a single composite tensor



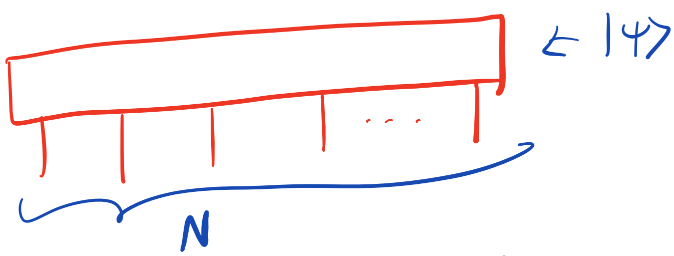
4 Quantum mechanics and tensor network

Multipartite quantum states are conveniently represented as TN.

$$|\psi\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d \otimes \cdots \mathbb{C}^d$$

e.g.: 

spin-d degree of freedom on a lattice



- Importantly, the splitting of this rank N tensor into N tensors reveal the entanglement structure of the state.

- ## Bipartite entanglement of pure states

Consider a pure quantum state on $\mathcal{H}_A \otimes \mathcal{H}_B$ represented as $|\psi\rangle = \sum_{ij} \psi_{ij} |a_i\rangle |b_j\rangle$ where $|a_i\rangle$ and $|b_j\rangle$ are an orthonormal basis for $\mathcal{H}_A$ and $\mathcal{H}_B$ respectively.

$|\psi\rangle$ is entangled if $|\psi\rangle \neq |\psi\rangle_A \otimes |\psi\rangle_B$.

example: $\quad |\psi\rangle = \dfrac{|00\rangle + |11\rangle}{\sqrt{2}} \neq |\psi\rangle_A \otimes |\psi\rangle_B$.

An information theoretic measure of pure state bipartite entanglement: von Neumann entropy of the reduced density matrix

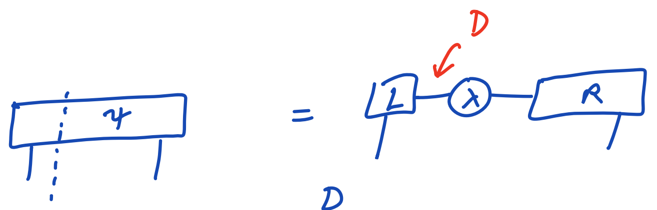
$$
\begin{aligned}
\rho_A &= \mathrm{tr}_B |\psi\rangle\langle\psi| \\
&= \sum_j \langle b_j|\psi\rangle\langle\psi|b_j\rangle \\
&= \sum_{i_1 i_2} \psi_{i_1 j} \psi^*_{i_2 j} |a_{i_1}\rangle\langle a_{i_2}|
\end{aligned}
$$

$$
S_{VN}(\rho_A) \equiv -\mathrm{tr}_A\, \rho_A \ln \rho_A = -\sum_i \mu_i \ln \mu_i
$$

suppose μ_i are the eigenvalues of ρ_A

μ_i must be real and non-negative (why?)

SVD provides a convenient way to estimate entanglement:



$(\Leftrightarrow \psi = U S V^\dagger)$

$$|\psi\rangle = \sum_i^D \lambda_i |L_i\rangle \otimes |R_i\rangle$$

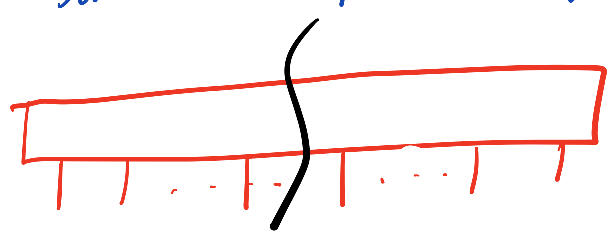
$|L_i\rangle$ and $|R_i\rangle$ are the columns of U and V matrices in SVD of ψ and λ_i are the singular values.

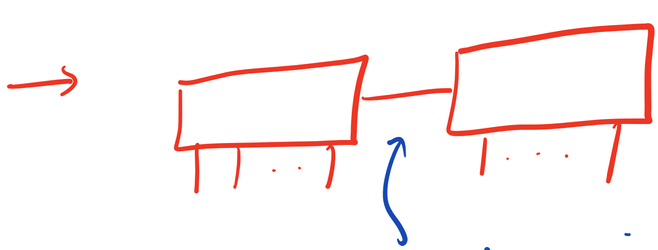
<u>Exercise</u>: Show $S_{VN}(\rho_2) = -\sum_i \lambda_i^2 \ln \lambda_i^2$

Say D is the Schmidt rank of the SVD.

An important property: $S_{VN} \leq \log D$

Recall the multipartite quantum state





the dimension of the virtual bond (D) indicates the entanglement in the state!