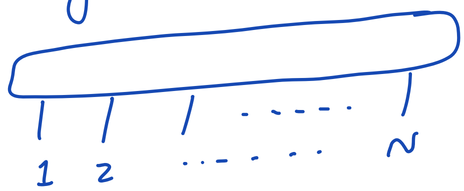
A. Matrix product states.

$$|\psi\rangle = \sum_{j_1, j_2, \cdots j_N = 0}^{d} \psi_{j_1 j_2 \cdots j_N} \, |j_1 j_2 \cdots j_N\rangle$$

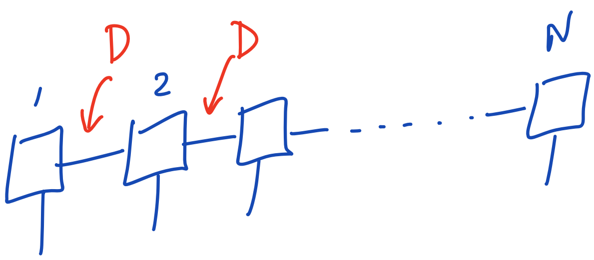
is a completely general state of N qudits

Diagrammatically, this can be represented as



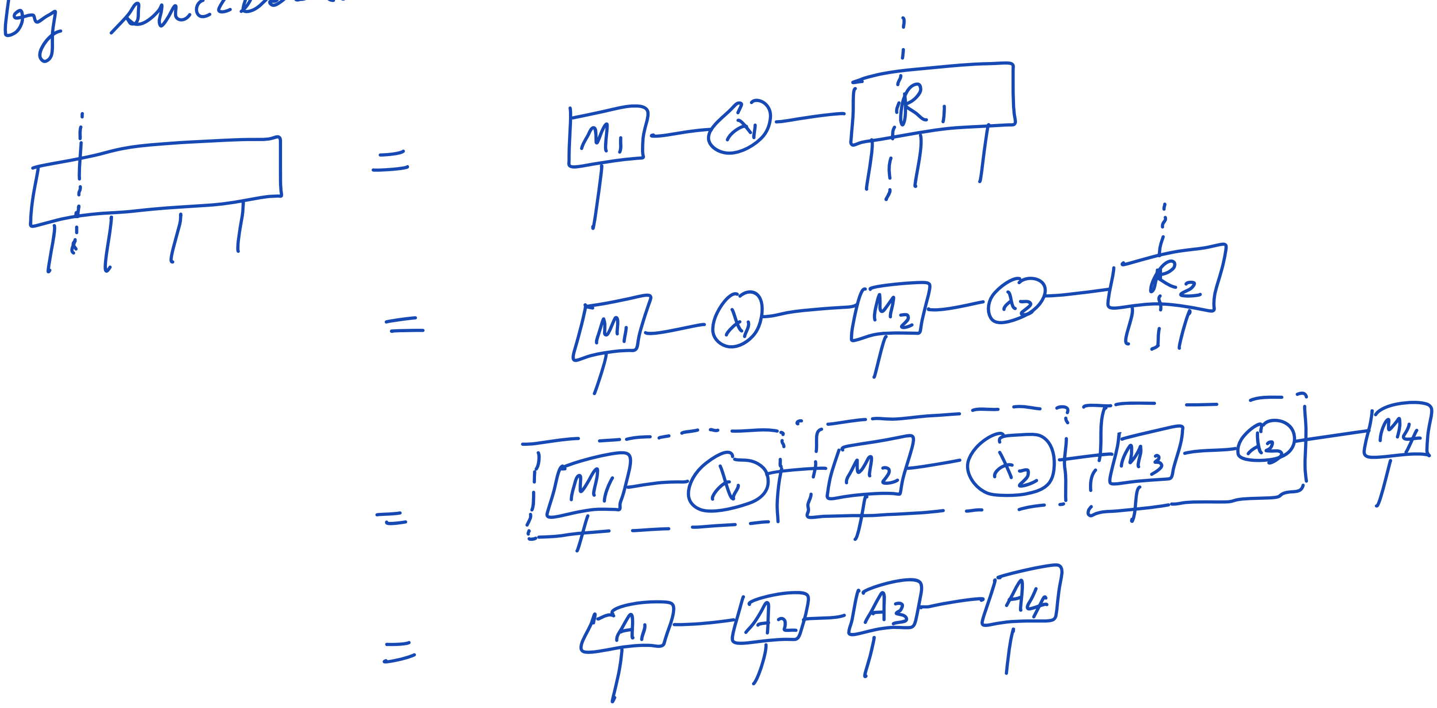
(d^N entries)

This can be converted into a TN diagram such as



($\sim dND^2$ entries)

by successive SVDs.



For a general quantum state on N qudits, the maximal bond dimension $D = d^{N/2}$ near the center of the chain.

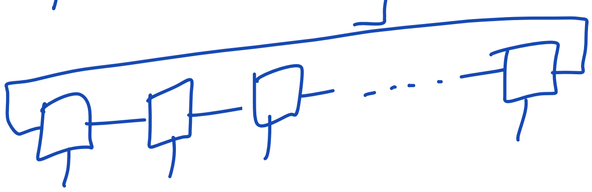
However, we can truncate $D \sim O(1)$ number and represent a class of slightly entangled states with a strong area law of entanglement, $S_0 \leq \log D$.

[Remark : if $D = d^{N/2}$, $S_0 \leq \frac{N}{2} \log d$, that scales with N, hence would be called volume law].

$$\left| \Psi \left[A^{(1)}, A^{(2)}, \ldots, A^{(N)} \right] \right\rangle$$

$$= \sum_{i_1 i_2 \ldots i_N} \mathrm{Tr}\left[A^{(1)}_{i_1} A^{(2)}_{i_2} \cdots A^{(N)}_{i_N} \right] |i_1 i_2 \ldots i_N\rangle$$ on periodic boundary condition

$\equiv$



$$\sum_{i_1 i_2 \ldots i_N} \langle v_L | A^{(1)}_{i_1} A^{(2)}_{i_2} \ldots A^{(N)}_{i_N} | v_R \rangle \, |i_1 \ldots i_N\rangle$$ on open boundary condition

Example:

(1) product state on qubits

$$\left.\begin{aligned} &|\psi\rangle = |0\cdots 0\rangle \\ &A_0 = (1) \qquad A_1 = (0) \end{aligned}\right\} D = 1$$

(2) GHZ state

$$\left.\begin{aligned} &|\psi\rangle = \frac{|0\cdots 0\rangle + |1\cdots 1\rangle}{\sqrt{2}} \\ &A_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\ &\langle v_L| = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \end{pmatrix} \\ &|v_R\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \end{aligned}\right\} D = 2$$

$\downarrow$

>1 implies entanglement

B Quantum phases of matter

Suppose you are given a microscopic Hamiltonian. What are the properties of its ground state in the thermodynamic limit?

Example: consider a 1d Lattice of spin $\frac{1}{2}$.

$$H = -\sum_i Z_i Z_{i+1} - g \sum_i X_i \qquad \left(\begin{array}{l} Z_i \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ X_i \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{array} \right)$$

Features :

1. Symmetry $= \prod_i X_i$
2. Locality

$g \gg 1$: $|gs\rangle \sim |+\rangle |+\rangle \cdots |+\rangle$

$g \ll 1$: $|gs\rangle \sim \dfrac{|00\ldots0\rangle \pm |1\ldots1\rangle}{\sqrt{2}}$

Ferromagnetic $\longrightarrow$ paramagnetic

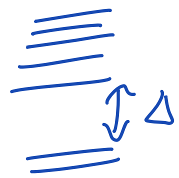
g

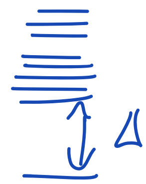
$\Downarrow$ $\Downarrow$

1. 2 degenerate ground state — 1 symmetric ground state

2. $\langle Z_i Z_{i+r} \rangle \sim O(1)$ — $\langle Z_i Z_{i+r} \rangle = e^{-r}$

3. Gapped





$\Delta \sim 2(1-|g|)$ — $\Delta \sim 2(|g|-1)$

4. $S_{VN} \sim$ area law — $S_{VN} \sim$ area law

At $g=1$ $\rightarrow$ energy gap closes

$\rightarrow$ ground state has power law correlations:

$C(r) \sim |r|^{-\nu}$

$\rightarrow$ Entanglement entropy for a region A

$S(A) \sim \log |A|$, where $|A|$ is the size of region.

Away from $g=1$, gapped phase of matter with decaying correlations and bounded entanglement

$\Rightarrow$ can be efficiently represented by MPS!

"Theorem": Ground states of (gapped) local Hamiltonians fulfill area law $S \sim L^{D-1}$
[proof exists in 1d]

1D area law $\Rightarrow$ Schmidt values decay quickly and thus we can find a good approx. of $|\psi\rangle$ by keeping D constant in an MPS.

C MPS properties

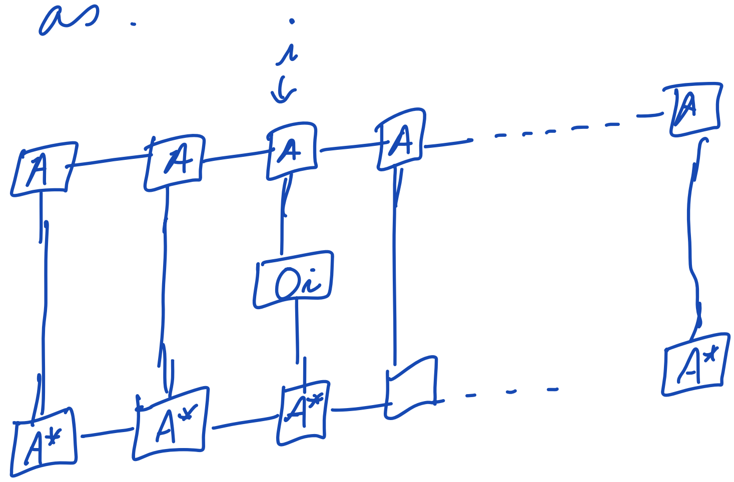
- For any MPS with max. bond dimension D, the entanglement is bounded by $\log D$.

- MPS with fixed bond dimension also generically demonstrate decay of correlations.

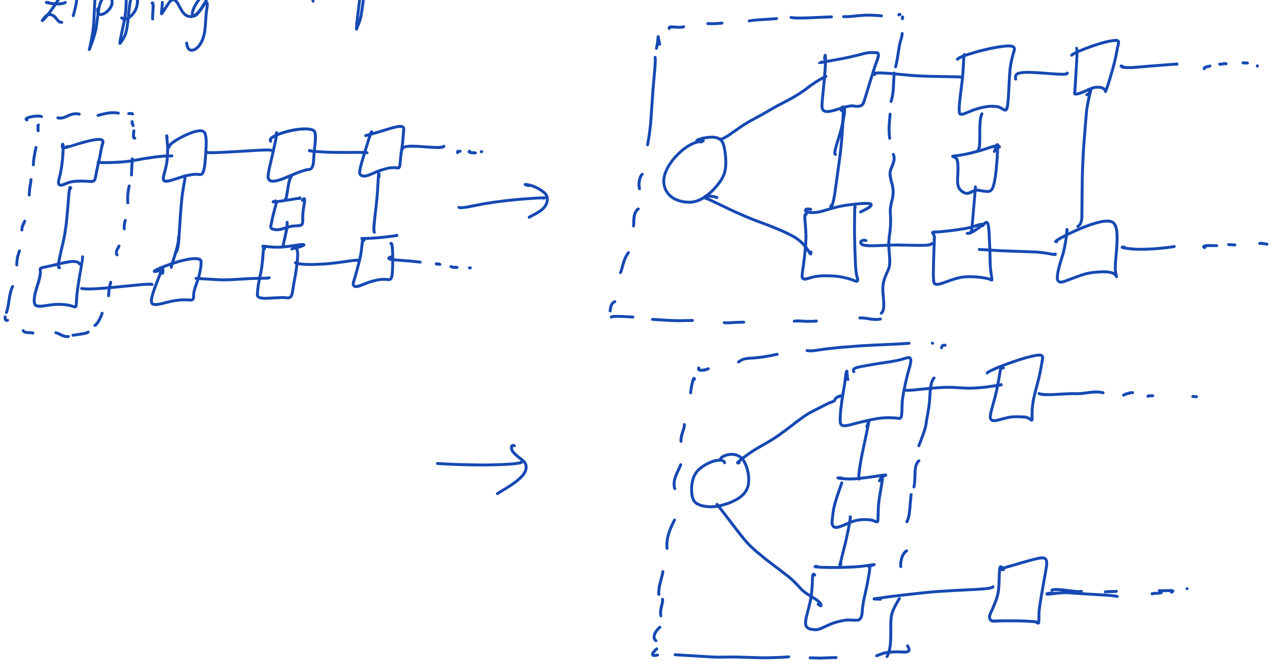
- Extraction of correlation functions using TN.

We are interested in computing expectation values of local operators $\langle \psi | O_i | \psi \rangle$.

This can be represented diagrammatically for an MPS as:

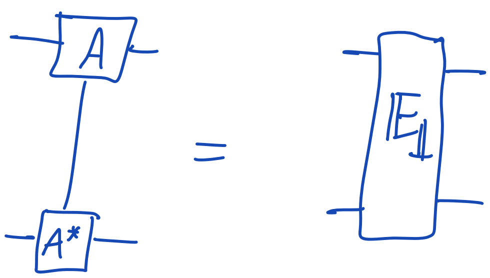


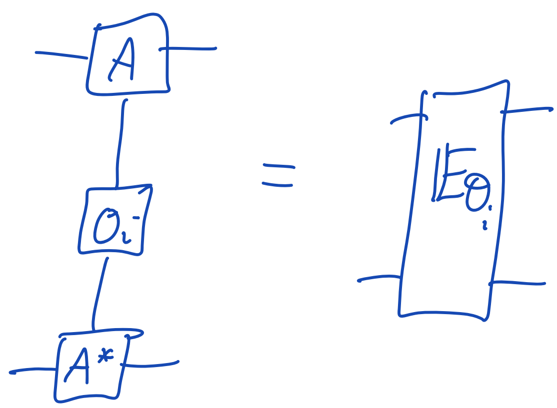
This can be computed efficiently by "zipping" up the ladder.



so on...

Transfer matrix





$$\therefore \langle O_i \rangle = \mathrm{tr}\left[(\mathbb{E}_1 \mathbb{E}_1 \cdots) \mathbb{E}_{O_i} (\mathbb{E}_1 \mathbb{E}_1 \cdots) \right]$$

We can analyze the thermodynamic properties of MPS by the spectral analysis of $\mathbb{E}_{\mathbb{1}}$

Let λ_i be the D^2 eigenvalues of $\mathbb{E}_{\mathbb{1}}$ in the decreasing order of magnitude $\lambda_1 \geq \lambda_2 \cdots \geq \lambda_{D^2}$

$$\mathbb{E}_{\mathbb{1}} = \lambda_1 \sum_{i=1}^{D^2} \left(\frac{\lambda_i}{\lambda_1}\right) \vec{R}_i^{\,T} \vec{L}_i$$

Right evectors

Left evectors.

Assume largest eigenvalue λ_1 is non-degenerate.

$$\therefore \mathbb{E}_{\mathbb{1}}^{\infty} \propto \vec{R}_1^{\,T} \vec{L}_1$$ (projector onto dominant eigenvector)

Suppose w be the degeneracy of λ_2.

Interested in 2 body correlator

$$C(r) = \langle O_i O_{i+r} \rangle - \langle O_i \rangle \langle O_{i+r} \rangle$$

$$= \frac{\left(tr\left[(\mathbb{E}_{\mathbb{1}})^{\infty} \mathbb{E}_{O_i} (\mathbb{E}_{\mathbb{1}})^{r-1} \mathbb{E}_{O_{i+r}} (\mathbb{E}_{\mathbb{1}}^{\infty}) \right] - tr\left[\mathbb{E}_{\mathbb{1}}^{\infty} \mathbb{E}_{O_i} \mathbb{E}_{\mathbb{1}}^{\infty} \right] tr\left[\mathbb{E}_{\mathbb{1}}^{\infty} \mathbb{E}_{O_{i+r}} \mathbb{E}_{\mathbb{1}}^{\infty} \right] \right)}{tr\left(\mathbb{E}_{\mathbb{1}}^{\infty} \right)}$$

$$\langle O_i O_{i+r}\rangle = \frac{\overbrace{(\vec{L}_1 E_{O_i} \vec{R}_1^T)(\vec{L}_1 E_{O_{i+r}} \vec{R}_1^T)}^{\langle O_i\rangle\langle O_{i+r}\rangle}}{\lambda_1^2}$$

$$+ \left(\frac{\lambda_2}{\lambda_1}\right)^{r-1} \sum_{\mu=2}^{\omega+1} \frac{(\vec{L}_1 E_{O_i} \vec{R}_\mu^T)(\vec{L}_\mu E_{O'} \vec{R}_1^T)}{\lambda_1^2}$$

$$\therefore \; C(r) \sim \left(\frac{\lambda_2}{\lambda_1}\right)^{r-1} \sim e^{-r/\xi}$$

$\rightarrow$ exponentially decaying with r.

If $\lambda_2 = 0$, $\langle O_i O_{i+r}\rangle \approx \lambda_1^2 \rightarrow$ constant.

Q: What kind of correlations do MPS states represent?