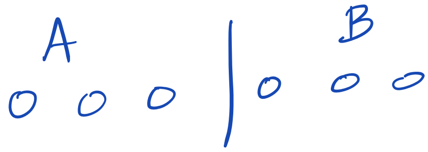
Recall :

- Entanglement :



$$|\psi\rangle \in \mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$$

$$\overset{\shortparallel}{\sum_{ij}} \psi_{ij} \, |i\rangle_A |j\rangle_B$$

$$\rho_A = \sum_{i i' j} \psi^*_{i'j} \, \psi_{ij} \, |i'\rangle\langle i|_A = tr_B(|\psi\rangle\langle\psi|)$$

Entanglement entropy : $S = - Tr_A \, \rho_A \log \rho_A$

- Schmidt decomposition : $\cong$ SVD

$$|\psi_A\rangle = \sum_{\alpha=1}^{\min(N_A, N_B)} \lambda_\alpha \, |\phi_\alpha\rangle_A \, |\phi_\alpha\rangle_B \quad \text{such that}$$

N_A: $\dim(\mathcal{H}_A)$

$$\langle \phi_\alpha | \phi_{\alpha'} \rangle = \delta_{\alpha\alpha'}$$

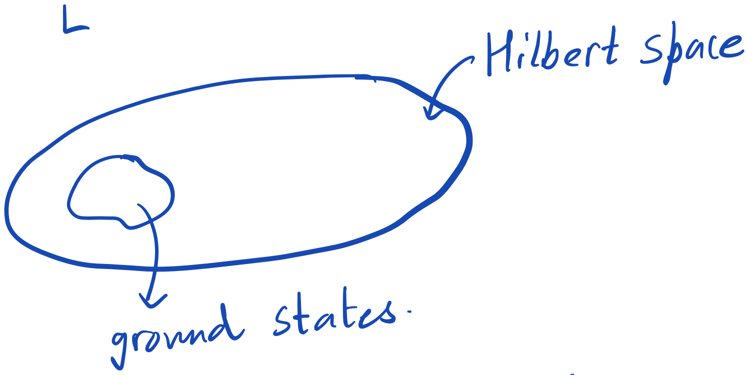
$$\rho_A = \sum_\alpha \lambda_\alpha^2 \, |\phi_\alpha\rangle_A \, {}_A\langle \phi_\alpha|$$

$$\Rightarrow S = - \sum_\alpha \lambda_\alpha^2 \log \lambda_\alpha^2$$

- Area law:

Ground states of (gapped) local Hamiltonians fulfill the area law $S \sim L^{D-1}$

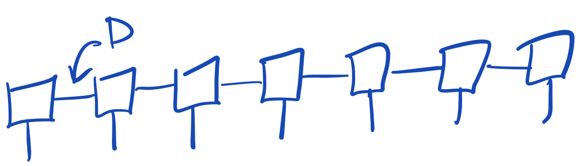
1D. $\underbrace{\overset{L/2}{\circ\ \circ\ \circ}\ \Big|\ \overset{L/2}{\circ\ \circ\ \circ}}_{L}$ $\qquad S(L) = \text{const.} \quad (L > \xi)$



1D area law $\Rightarrow$ Schmidt values decay quickly and thus we can approximate $|\psi\rangle$ by keeping $D = \text{const}$, with Schmidt decomposition

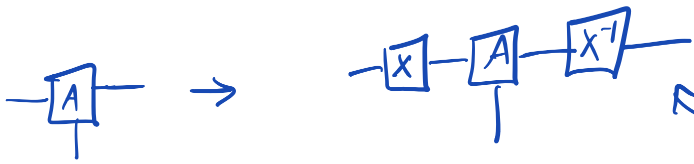
$$|\psi\rangle \simeq \sum_{\alpha=1}^{D} \lambda_\alpha |\alpha\rangle_A |\alpha\rangle_B$$

- Matrix product states: constant bond dimension D



Canonical form of MPS

MPS are not uniquely defined.

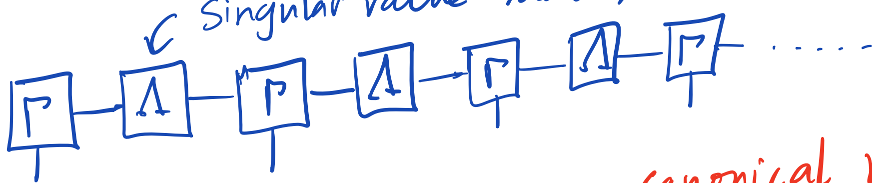


represents same state.

Bonds are directly related to Schmidt decomposition.

$$|\psi\rangle \overset{SVD}{=} \sum_{\alpha} \lambda_{\alpha} |\alpha\rangle_L |\alpha\rangle_R$$

Singular value matrix

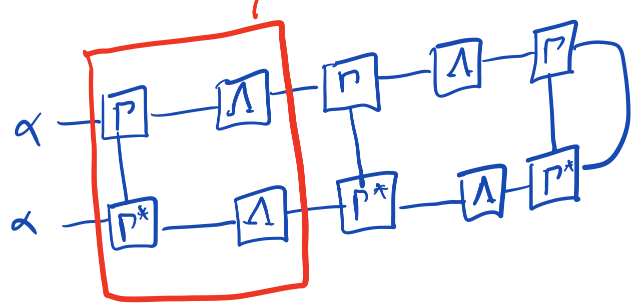
$=$ 

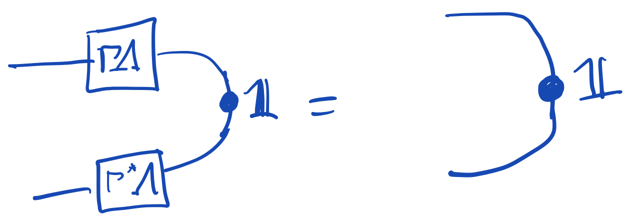
this is a canonical representation.

Because of SVD : orthonormality of "Right" vectors

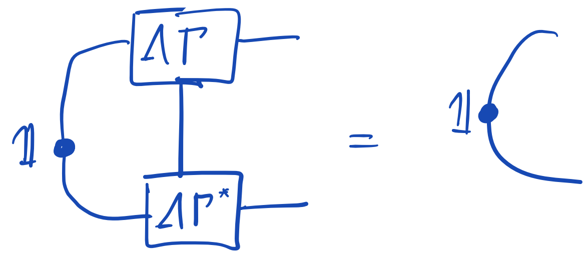
$$\delta_{\alpha\alpha'} = {}_R\langle \alpha | \alpha' \rangle_R$$

Transfer matrix

$=$ 

$=$ 

[Similar for the left] $\rightsquigarrow$

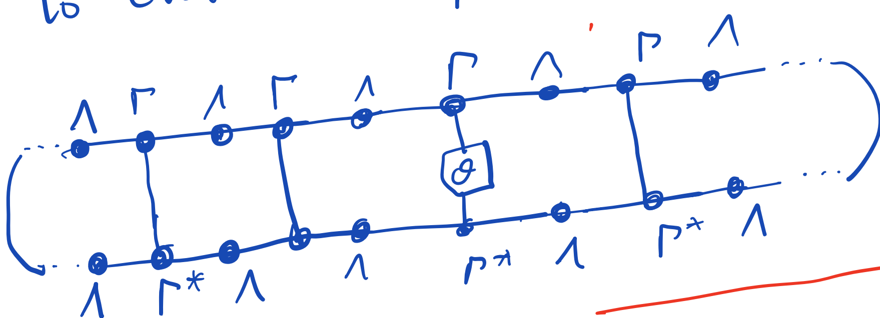


$\Leftrightarrow$ Transfer matrices have left/right eigenvalue 1 with eigenvector $\mathbb{1}$

$\Leftrightarrow$ uniquely defines MPS upto a $U(1)$ phase $\sim e^{i\theta}$, that does not change the global state.

Very convenient to evaluate expectation values:

$\langle\psi| O_i |\psi\rangle =$



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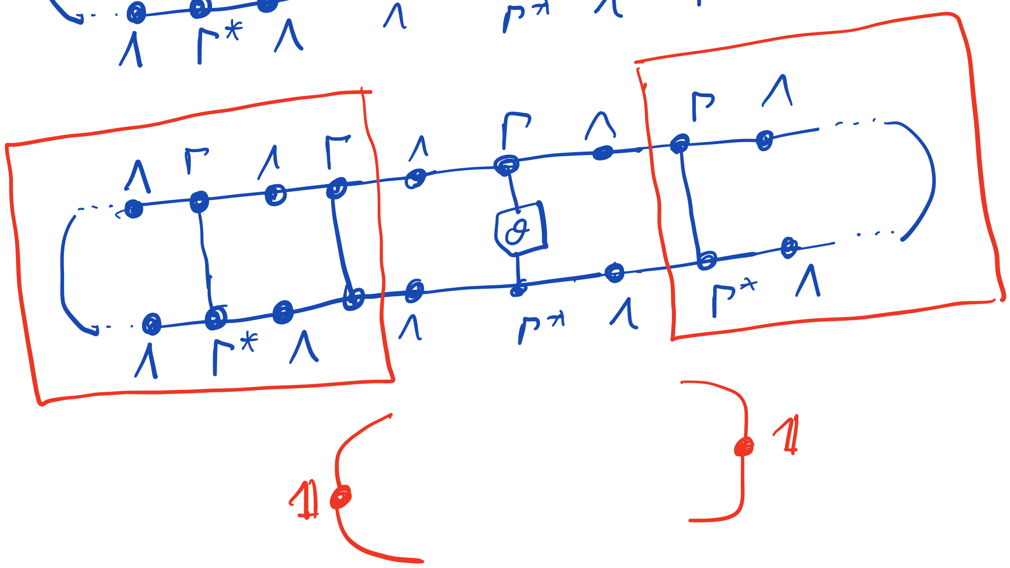

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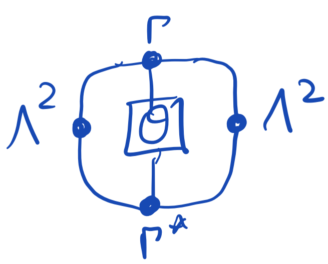


$\rightarrow$ Easy to compute!

Time Evolving Block Decimation (TEBD) [Vidal '03]

Algorithm to find ground state given a local Hamiltonian H.

Time evolution in imaginary time yields the ground state.

Start from an initial state $|\psi_i\rangle$

$$|\psi_i\rangle = \sum_\alpha \psi_i^\alpha \, |\alpha\rangle$$

↑ energy eigenstate.

$$H|\alpha\rangle = \epsilon_\alpha |\alpha\rangle \; ; \; \epsilon_1 \leq \epsilon_2 \leq \epsilon_3 \leq \cdots$$

↑ energy ↑ ground state.

$$e^{-\tau H} |\psi_i\rangle = \sum_\alpha (\psi_i^\alpha) \, e^{-\tau \epsilon_\alpha} |\alpha\rangle$$

If we take $\tau \to \infty$, only the lowest energy will be remaining: $e^{-\tau H}|\psi_i\rangle \xrightarrow{\tau\to\infty} \underbrace{\psi_i^{[\alpha=1]}}_{\text{this shouldn't be zero}} e^{-\tau\epsilon_\alpha} \underbrace{|\alpha=1\rangle}_{\text{ground state}}$

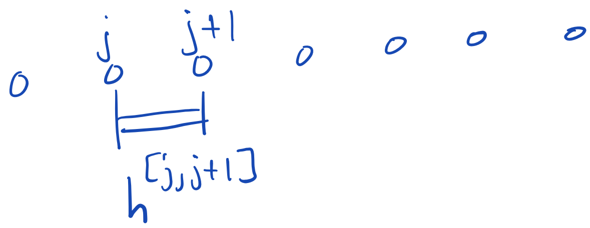
$$|gs\rangle = \lim_{\tau\to\infty} \frac{e^{-H\tau}|\psi_i\rangle}{\| e^{-H\tau}|\psi_i\rangle \|}$$

"Trotterization"

Assume the Hamiltonian has the form $H = \sum_j h^{[j,j+1]}$

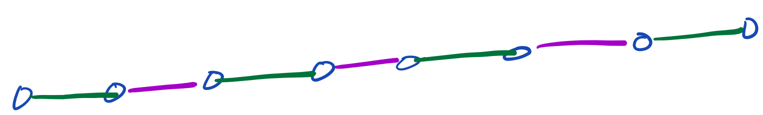
(Recall the Transverse field Ising model:

$$H = -\sum_j Z_j Z_{j+1} - g \sum_i X_i$$
)



Decompose the Hamiltonian $H = F + G$

$$F = \sum_{\text{even } j} h^{[j,j+1]} \; ; \quad G = \sum_{\text{odd } j} h^{[j,j+1]}$$



Observe $[F^i, F^j] = [G^i, G^j] = 0$

but $[G, F] \neq 0$

$$e^{\varepsilon(A+B)} \neq e^{\varepsilon A} \cdot e^{\varepsilon B} \quad \text{if } [A,B] \neq 0$$

Baker Campbell Hausdorff formula

$$e^{\varepsilon A} \cdot e^{\varepsilon B} = e^{\varepsilon(A+B) + \frac{\varepsilon^2}{2}[A,B] + \cdots}$$

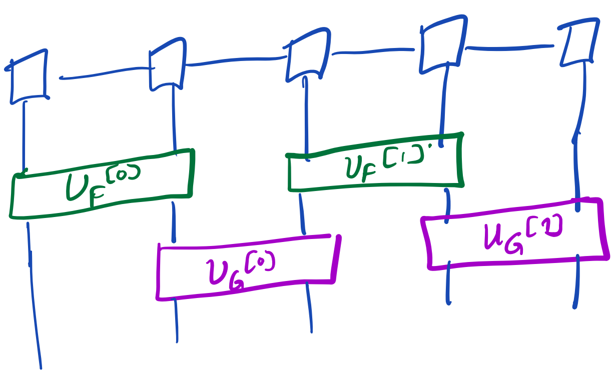
Decompose time evolution :

$$\exp(-iHt) = \left[\exp\left(-iH \underbrace{\frac{t}{N}}_{=\delta t}\right)\right]^N$$

$$e^{-i\delta t\,(F+G)} \approx \underbrace{e^{-i\delta t F}}_{U_F}\ \underbrace{e^{-i\delta t\, G}}_{U_G} + \mathcal{O}(\delta t^2)$$

$$U_F = \prod_{\text{even } j} e^{-iF^{[i]}\delta t} \; ; \quad U_G = \prod_{\text{odd } j} e^{-iG^{[i]}\delta t}$$

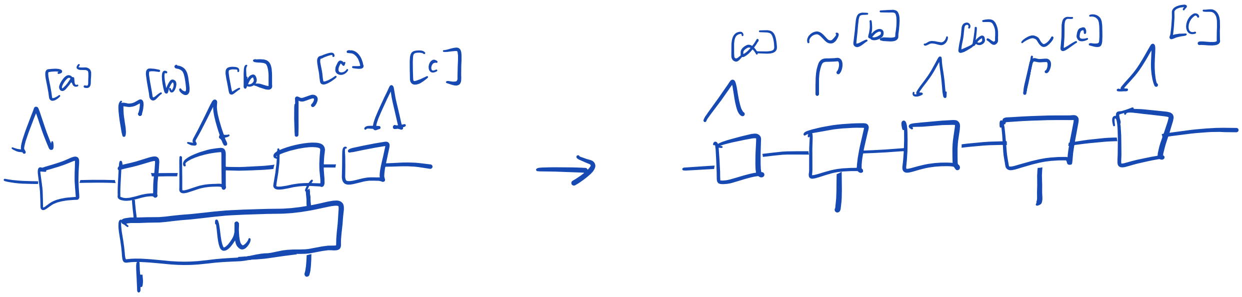
Evolution of an MPS for one time step :



$\rightarrow |\psi(0)\rangle$

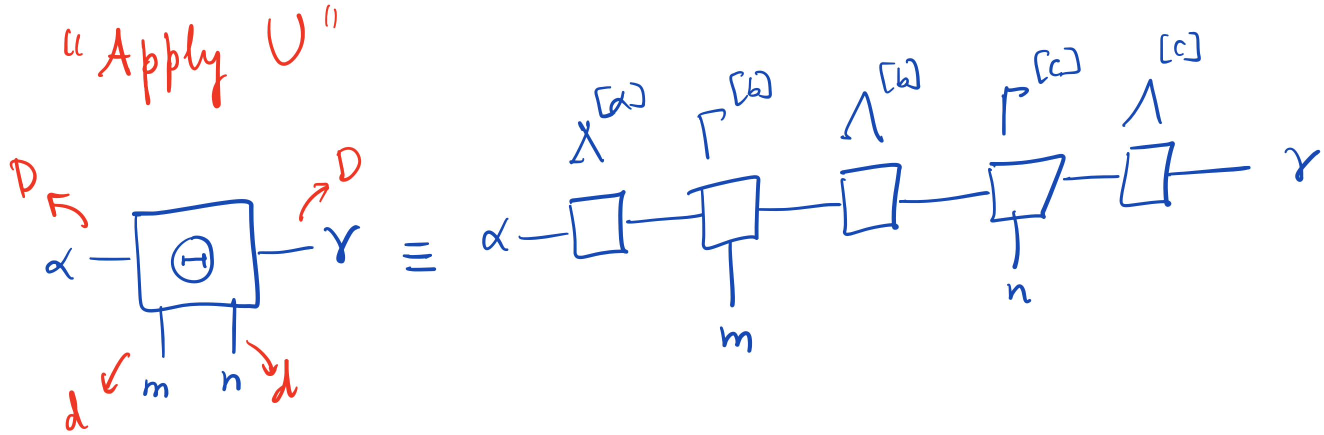
$\rightsquigarrow |\psi(\delta t)\rangle$

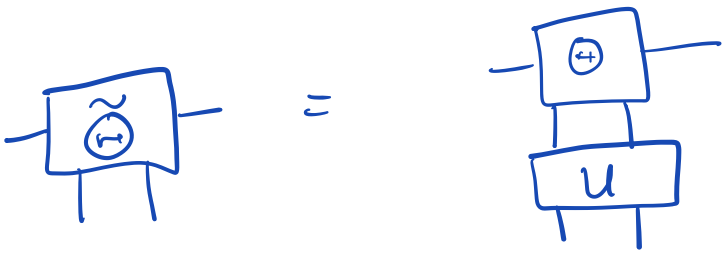
Need an algorithm to project back to MPS form



TEBD algorithm

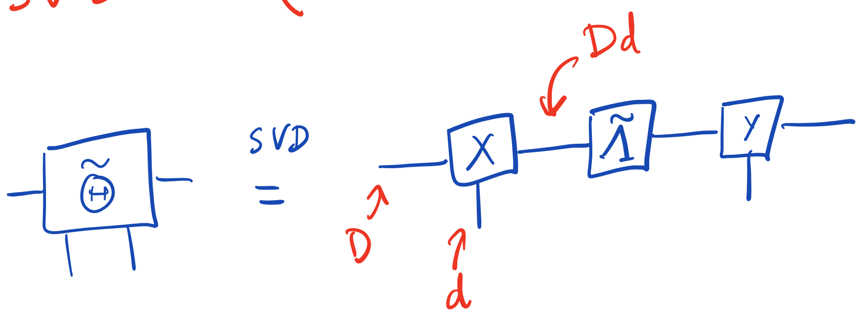
1. "Apply U"



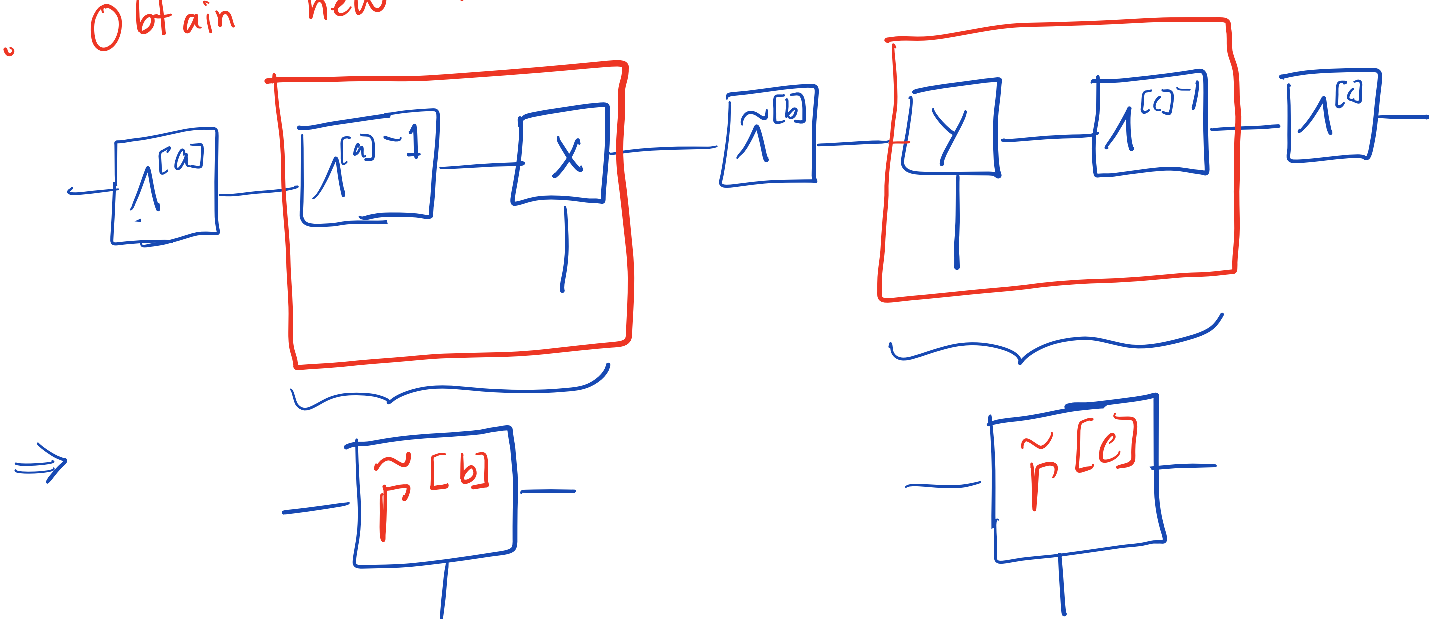


$$\left(\tilde{\Theta}^{mn}_{\alpha\gamma} = U^{mn}_{m'n'} \, \Theta^{m'n'}_{\alpha\gamma}\right)$$

2. "SVD" (dD x dD matrix)



3. Obtain new MPS.



4. Truncate

Discard smallest Schmidt values

$$Dd \longrightarrow D$$

(i.e keep only D rows/columns of tensors)

Applying this algorithm iteratively to even/odd bonds: we obtain the time evolution!

Computational time scales as $\mathcal{O}(L d^3 D^3)$

Errors:

1. Truncation error → take larger D
2. Trotter error → take smaller δt